$$\sum_{j=1}^{n} v_{ij} y_j^* < \sum_{j=1}^{n} v_{rj} y_j^* \quad \text{or} \quad E(\mathbf{e_i}, \mathbf{y}^*) < E(\mathbf{e_r}, \mathbf{y}^*)$$

$$\therefore \qquad v \geq E(\mathbf{e_r}, \mathbf{y}^*) > E(\mathbf{e_i}, \mathbf{y}^*) \qquad ...(2)$$

Now let $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_m^*)$ be an optimal strategy for the row player. If possible, let us suppose $x_i^* > 0$, then from (2) $\quad x_i^* v > x_i^* E(\mathbf{e_i}, \mathbf{y}^*)$

Also, we have
$$v = E(\mathbf{x}^*, \mathbf{y}^*) = \sum_{i=1}^{m} x_i^* E(\mathbf{e_i}, \mathbf{y}^*)$$

$$= x_r^* E(\mathbf{e_r}, \mathbf{y}^*) + \sum_{i \neq r}^{m} x_i^* E(\mathbf{e_i}, \mathbf{y}^*)$$

$$< x_r^* v + \sum_{i \neq r}^{m} x_i^* v = v \sum_{i=1}^{m} x_i^* = v \qquad \left(\because \sum_{i=1}^{m} x_i^* = 1\right)$$

which is a contradiction, and hence $x_i^* = 0$.

Second part can also be proved similarly.

19.14-1 Generalized Dominance Property

The dominance property is not only based on the superiority of pure strategies only, but on the superiority of some *convex linear combination* of two or more pure strategies also. A given strategy can also be said to be dominated if it is inferior to some convex linear combination of two or more strategies. This concept generalizes the above dominance principle in the following theorem.

Theorem. 19·5 (Generlized Dominance). *Let* $\mathbf{A} = [v_{ij}]$ *be the pay-off matrix of an* $m \times n$ *rectangular game. If the ith row of* **A** *is strictly dominated by a convex combination of the other rows of* **A**, *then the deletion of the ith row of* **A** *does not effect the set of optimal strategies for the row player (the player A).*

Further, if the jth column of **A** *strictly dominates a convex combination of the other columns, then the deletion of the jth column of* **A** *does not effect the optimal strategies for the column player (the player B).*

Proof. Let $A = [v_{ij}]$ be the payoff matrix considering the first part, we are given that there exist scalars (probabilities) $x_1, x_2, \ldots, x_m$ $(0 \leq x_i \leq 1, x_r = 0, \Sigma x_i = 1)$ such that

$$\sum_{i=1, i \neq r}^{m} x_i v_{ij} \geq v_{rj}, \text{ for } j = 1, 2, \ldots, n$$

or
$$\sum_{i=1}^{m} x_i v_{ij} \geq v_{rj}, \text{ for } j = 1, 2, \ldots n \qquad (\because x_r = 0) \qquad ....(1)$$

where strict inequality holds for at least one j.

Let $\mathbf{y}^* = (y_1^*, y_2^*, \ldots, y_n^*)$ be an optimal strategy for player B. Then it follows from (1) that

$$\sum_{j=1}^{n} v_{rj} y_j^* < \sum_{j=1}^{n} \sum_{i=1}^{m} v_{ij} x_i y_j^*,$$

or
$$E(\mathbf{e_r}, \mathbf{y}^*) < \sum_{j=1}^{n} \sum_{i=1}^{m} v_{ij} x_i y_j^* \leq v \qquad ...(2)$$

Let $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_m^*)$ be an optimal strategy for player A.

If possible, let us suppose that $x_r^* \neq 0$. From (2), we know that $E(\mathbf{e_r}, \mathbf{y}^*) < v$.

Then since $x_r^* \neq 0$, we must have $x_r^* E(\mathbf{e_r}, \mathbf{y}^*) < x_i^* v$.

Thus
$$E(\mathbf{x}^*, \mathbf{y}^*) = \sum_{j=1}^{n} \sum_{i=1}^{m} v_{ij} x_i^* y_j^* = x_r^* E(\mathbf{e_r}, \mathbf{y}^*) + \sum_{i \neq r} x_i^* E(\mathbf{e_i}, \mathbf{y}^*)$$

This implies
$$v < x_r^* v + v \sum_{i \neq r} x_i^* = v \sum_{i=1}^{m} x_i^* = v \text{ which is a contradiction.}$$

Hence we must have $x_r^* = 0$. This completes the proof for the first part.

Similarly, we can prove the second part.

Remarks:

1. If $v_{rj} = \sum_{1}^{m} x_i v_{ij}$ for all $j = 1, 2, ..., n$; the result follows trivially, for then any probability assigned to the rth row can be easily distributed over the other rows, and rth row itself is ignored.
2. It should also be noted here that the dominating column is deleted whereas the row dominated by a convex combination of other rows is deleted.

19·14-2. Summary of Dominance Rules

The *"dominance property"* can be summerized in the following rules :

Rule 1. If each element in one row, say rth of the payoff matrix $[v_{ij}]$, is less than or equal to the corresponding element in the other row, say sth, then the player A will never choose the rth strategy. In other words, if for all $j = 1, 2, ..., n$, and $v_{rj} \le v_{sj}$, then the probability x_r of choosing rth strategy will be zero. The value of the game and the non-zero choice of probabilities remain unaltered even if rth row is deleted from the payoff matrix. Such rth row is said to be dominated by the sth row.

Rule 2. Following the similar arguments, if each element in one column, say C_r, is greater than or equal to the corresponding element in the other column, say C_s, then the player B will never use the stategy corresponding to column C_r. In this case, the column C_s dominates the column C_r.

Rule 3. Dominance need not be based on the superiority of pure strategies only. A given strategy can be dominated if it is inferior to an average of two or more other pure strategies. In general, if some convex linear combination of some rows dominates the ith row, then the ith row will be deleted. If the ith row dominates the convex linear combination of some other rows, then one of the rows involving in the combination may be deleted. Similar arguments follow for columns also.

Rule 4. If (x_1, x_2) be the optimal strategy for the player A for the reduced game and (w_1, x_2) be the optimal strategy for the original game, then w_1 is the ith place extension of x_1.

Rule 5. If (y_1, y_2) be the optimal strategy for the player B for the reduced game and (y_1, w_2) be the optimal strategy for the original game, then w_2 is the jth place extension of y_2.

Rule 6. If the dominance holds strictly, then values of optimal strategies do coincide, and when the dominance does not hold strictly, then optimal strategies may not coincide.

Note. Using dominance properties, try to reduce the size of payoff matrix.

19·14-3. Demonstration of Dominance Properties by Examples

1. *To illustrate first and second properties,* consider the example of (3×3) game [*Table* 19·11].

It is clear that this game has no saddle point. However, consider Ist and IIIrd columns from player B's point of view. It is seen that each payoff (element) in IIIrd column is greater than the corrresponding element in Ist column regardless of the player A's strategy. Evidently, the choice of IIIrd strategy by the player B will always result in the greater loss as compared to that of selecting the Ist strategy. The column III is *inferior* to I as never to be used. Hence, deleting the IIIrd column which is dominated by I, the reduced-size payoff matrix (*Table 19·12*) is obtained.

Again, if the reduced matrix (*Table 19·12*) is looked from player A's point of view, it is seen that the player A will never use the II strategy which is dominated by III. Hence, the size of matrix can be reduced further by deleting the II row (*Table 19·12*). This reduced matrix can be further reduced by deleting II row as shown in *Table 19·13.* The solution of the reduced 2×2 matrix game without saddle point can be easily obtained by solving the following simultaneous equations in usual notations :

Table 19·11

		B: I	II	III
A	I	−4	6	3
	II	−3	−3	4
	III	2	−3	4

Table 19·12

		B: I	II
A	I	−4	6
	II	−3	−3
	III	2	−3

Table 19·13

		B: I	II
A	I	−4	6
	III	2	−3

$$-4x_1 + 2x_3 = v,\ 6x_1 - 3x_3 = v,\ x_3 + x_1 = 1 \text{ (For } A\text{)}$$

and $$-4y_1 + 6y_2 = v,\ 2y_1 - 3y_2 = v,\ y_1 + y_2 = 1 \text{ (For } B\text{)}$$

It is advisable to verify the solution :

(i) The player A chooses mixed strategy $(x_1, x_2, x_3) = (1/3, 0, 2/3)$

(ii) The player B chooses mixed strategy $(y_1, y_2, y_3) = (3/5, 2/5, 0)$

(iii) The value of the game is zero, *i.e.*, the game is fair.

Q. 1. Explain the term 'saddle point' and 'dominance' used in game theory. **[Raj. Univ. (M. Phil) 89]**

2. Write short note on 'concept of dominance'. **[Kanpur M.Sc (Math.) 93]**

3. Explain the concept of generalized dominancy in the context of game theory. **[Kanpur M.Sc. (Math.) 97; Delhi M.Sc. (Stat.) 95]**

4. Briefly explain the general rules for dominance. **[JNTU (B. Tech.) 2004. 03]**

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2. ***To illustrate the third property***, consider the following game matrix [*Table 19·14*]

None of the *pure* strategies of the player A is inferior to any of his other pure strategies. However, the average of the player A's first and second pure strategies gives

Table 19·14

A \ B	1	2	3
1	5	0	2
2	–1	8	6
3	1	2	3

$$\left\{\frac{5-1}{2}, \frac{0+8}{2}, \frac{2+6}{2}\right\} \quad \text{or} \quad (2, 4, 4)$$

Obviously, this is superior to the player A's third pure strategy. So the third strategy may be deleted from the matrix. The reduced matrix is shown in *Table* 19·15.

Table 19·15

	1	2	3
1	5	0	2
2	–1	8	6

Q. 1. Explain the following terms :
(i) Two-person zero-sum game, (ii) Principle of dominance, (iii) Pure strategy in game theory. **[Agra 94]**

2. How is the concept of dominance used in simplifying the solution of a rectangular game ? **[Meerut (OR) 2003]**

3. Explain the principle and rules of dominance to reduce the size of payoff matrix. **[VTU (BE Mech.) 2002]**

4. State the general rules of dominance for two person zero-sum games.

5. Let R_1, R_2 be the subsets of the rows of an $m \times n$ payoff matrix A. Likewise, let C_1, C_2 be the subsets of the columns of A. Show that if a convex combination of the rows (columns) in R_1 (C_1) dominates a convex combination of the rows (columns) in R_2 (C_2), Then there exists a row (column) in R_2 (C_1) which, if deleted, does not change the set of optimal strategies for player A (player B).

6. Show that the existence of a saddle point in 2×2 game implies the existence of a dominating pure strategy for at least one of the players and conversely.

Solved Examples

Example 9. *Solve the game whose payoff matrix to the player A is given in the table :*

A \ B	I	II	III
I	1	7	2
II	6	2	7
III	5	2	6

[Rohil 91]

Solution. Since the row III is inferior to the row II, row III can be deleted from the payoff matrix. Thus the reduced matrix (*Table 19·16*) is obtained.

Table 19·16

A \ B	I	II	III
I	1	7	2
II	6	2	7

Again, column III is dominated by column I, therefore column III can also be deleted from the above matrix. The reduced matrix is given in (*Table 19·17*).

Table 19·17

A \ B	(y_1) I	(y_2) II
(x_1) I	1	7
(x_2) II	6	2

This 2×2 game without saddle point can be solved either by putting $v_{11} = 1$, $v_{12} = 7$, $v_{21} = 6$, $v_{22} = 2$ in the formulae of *Sec.* **19·13,** or by solving the simultaneous equations :

$1x_1 + 6x_2 = v$, $7x_1 + 2x_2 = v$, $x_1 + x_2 = 1$ ***(For player A)***

$1y_1 + 7y_2 = v$, $6y_1 + 2y_2 = v$, $y_1 + y_2 = 1$ ***(For player B)***

Thus the following solution is obtained :

(i) The player A chooses optimal strategy $(x_1, x_2, x_3) = (2/5, 3/5, 0)$.

(ii) The player B chooses optimal strategy $(y_1, y_2, y_3) = (1/2, 1/2, 0)$.

(iii) The value of the game to the player A is $v = 4$.

Example 10. *Use the relation of dominance to solve the rectangular game whose payoff matrix to A is given in Table 19·18.* **[Raj. (M.Phil) 92; Delhi B.Sc. (Math) 93; I.C.W.A. (June) 92]**

Table 19·18

A \ B	I	II	III	IV	V	VI
I	0	0	0	0	0	0
II	4	2	0	2	1	1
III	4	3	1	3	2	2
IV	4	3	7	−5	1	2
V	4	3	4	−1	2	2
VI	4	3	3	−2	2	2

Solution. In the payoff matrix from player A's point of view, rows I and II are dominated by the row III. Hence the player A will never use strategies I and II in comparison to the strategy III. Thus, deleting I and II rows we obtain the reduced matrix (*Table 19·19*).

Table 19·19

	I	II	III	IV	V	VI
III	4	3	1	3	2	2
IV	4	3	7	−5	1	2
V	4	3	4	−1	2	2
VI	4	3	3	−2	2	2

Again, from the player B's point of view, columns I, II and VI are dominated by the column V. Therefore, the player B will never use strategies I, II and VI in comparison to the strategy V. Now, delete columns I, II and VI from the matrix to obtain the new matrix (*Table 19·20*).

Table 19·20

A \ B	III	IV	V
III	1	3	2
IV	7	−5	1
V	4	−1	2
VI	3	−2	2

Again the row VI is dominated by the row V from the player A's point of view. Hence, deleting VIth row, obtain the next reduced matrix. [*Table 19·21*]

Table 19·21

	III	IV	V
III	1	3	2
IV	7	−5	1
V	4	−1	2

None of the pure strategies of the player B is inferior to any of his other strategies. However, the average of player B's III and IV pure strategies gives,

$$\left\{\frac{1+3}{2}, \frac{7-5}{2}, \frac{4-1}{2}\right\} \text{ or } (2, 1, 3/2)$$

which is obviously superior to the player B's Vth pure strategy, because Vth strategy will result much more losses to B. Thus deleting the Vth strategy from the matrix, the revised matrix (*Table 19·22*). is obtained :

Table 19·22

A \ B	III	IV
III	1	3
IV	7	−5
V	4	−1

Also, the average of the player A's III and IV pure strategies give

$$\left\{\frac{1+7}{2}, \frac{3-5}{2}\right\} \text{ or } (4, -1).$$

This is obviously the same as the player A's Vth strategy.

In this case, the Vth strategy may be deleted from the matrix. Finally, (2×2) reduced matrix (*Table 19·23*) is obtained.

Table 19·23

A \ B	(y_3) III	(y_4) IV
(x_3) III	1	3
(x_4) IV	7	−5

Now, for (2×2) game, having no saddle point, solve the following simultaneous equations :

$1.x_3 + 7x_4 = v$, $3x_3 - 5x_4 = v$, $x_3 + x_4 = 1$ (*For A*)

$1.y_3 + 3y_4 = v$, $7y_3 - 5y_4 = v$, $y_3 + y_4 = 1$ (*For B*)

The solution is :

(i) The player A chooses the optimal strategy $(0, 0, 6/7, 1/7, 0, 0)$.

(ii) The player B chooses the optimal strategy $(0, 0, 4/7, 3/7, 0, 0)$.

(iii) The value of the game to player A is $13/7$.

Example 11. *Two competitors A and B are competing for the same product. Their different strategies are given in the following payoff matrix:*

Table 19·24 (a)

Company B

		I	II	III	IV
Company A	I	3	2	4	0
	II	3	4	2	4
	III	4	2	4	0
	IV	0	4	0	8

use dominance principle to find the optimal strategies. **[JNTU (B. Tech.) 2003 Type; Meerut 02; Delhi (Stat.) 95, B.Sc. (Math.) 90; Rohilkhand 94, 93; Banasthali 93; Kanpur 93]**

Solution. First, we can find that this game does not have a saddle point. Now try to reduce the size of the given payoff matrix by using the principle of dominance.

From the player A's point of view, Ist row is dominated by the IIIrd row. So delete Ist row from the matrix [*Table 19·24 (a)*].

Again, from the player B's point of view, 1st column is dominated by the IIIrd column. Hence, Ist column may also be deleted from the matrix. Thus, the reduced payoff matrix [*Table 19·24 (b)*] is obtained.

Table 19·24 (b)

B

		II	III	IV
A	II	4	2	4
	III	2	4	0
	IV	4	0	8

In order to check the further reduction of this reduced matrix, the average of the player B's III and IV pure strategies give

$$\left\{\frac{2+4}{2}, \frac{4+0}{2}, \frac{0+8}{2}\right\} \text{ or } (3, 2, 4)$$

which is obviously superior to the player B's II pure strategy. Under this condition, the player B will not use II strategy. Hence, II column may be deleted from the matrix. Thus, new matrix (*Table 19·25*), is obtained.

Table 19·25

B

		III	IV
A	II	2	4
	III	4	0
	IV	0	8

Again, in the new matrix, the average of the player A's III and IV pure strategies give

$$\left\{\frac{4+0}{2}, \frac{0+8}{2}\right\} \text{ or } (2, 4)$$

which is obviously the same as the player A's II strategy. Therefore, the player A will gain the same amount even if the II strategy is never used by him. Hence deleting the player A's II strategy from the matrix to obtain the reduced (2×2) matrix (*Table 19·26*).

Table 19·26

B

			(y_3) III	(y_4) IV
A	x_3	III	4	0
	x_4	IV	0	8

Since this (2×2) payoff matrix has no saddle point, solve the simultaneous equations :

$4.x_3 + 0x_4 = v$, $0x_3 + 8x_3 = v$, $x_3 + x_4 = 1$ (*For player A*)

$4.y_3 + 0y_4 = v$, $0y_3 + 8y_4 = v$, $y_3 + y_4 = 1$ (*For player B*)

to get the solution :

(i) Optimal strategy for the player $A = (x_1, x_2, x_3, x_4) - (0, 0, 2/3, 1/3)$.

(ii) Optimal strategy for the player $B = (y_1, y_2, y_3, y_4) = (0, 0, 2/3, 1/3)$.

(iii) The value of the game to the player A is $v = 8/3$.

Example 12. *Solve the following game using dominance principle :*

Player B

$$\begin{array}{ll} & \begin{array}{ccccc} \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} \end{array} \\ \begin{array}{r} \text{I} \\ \text{Player } A \quad \text{II} \\ \text{III} \\ \text{IV} \end{array} & \begin{bmatrix} 3 & 5 & 4 & 9 & 6 \\ 5 & 6 & 3 & 7 & 8 \\ 8 & 7 & 9 & 8 & 7 \\ 4 & 2 & 8 & 5 & 3 \end{bmatrix} \end{array}$$

Solution. In the given payoff matrix, IVth column dominates the Ist column and also Vth column dominates the IInd column. So Ist and IInd columns can be deleted without affecting the optimal strategies of B. Thus we get the reduced payoff matrix (a).

Again, we observe that IIIrd row of the reduced matrix dominates all the other rows. Thus the payoff matrix (b) is obtained.

Again, the IInd column of (b) is dominated by both the Ist and IIIrd columns. Thus the reduced payoff matrix (c) is obtained.

$$\begin{array}{cc} & B \\ & \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ A \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \end{array} & \left[\begin{array}{ccc} 3 & 5 & 4 \\ 5 & 6 & 3 \\ 8 & 7 & 9 \\ 4 & 2 & 8 \end{array}\right] \end{array} \qquad \begin{array}{cc} & \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \text{III} & \begin{array}{|c|c|c|} \hline 8 & 7 & 9 \\ \hline \end{array} \end{array} \qquad \begin{array}{cc} & \text{II} \\ \text{III} & \begin{array}{|c|} \hline 7 \\ \hline \end{array} \end{array}$$

(a) (b) (c)

Thus the solution to the game is :

(i) best strategy for player A is III; *(ii)* best strategy for player B is II; and

(iii) value of the game for player A is 7, and for player B is – 7.

Note. If we apply principle of dominance to the payoff matrix having a saddle point, then we get a single element reduced matrix only. So the students are advised to use the principle of dominance for solving the games without saddle point until unless otherwise stated.

EXAMINATION PROBLEMS

Explain the principle of dominance and hence solve the following games :

1.

$$\begin{array}{cc} & \text{Player } B \\ & \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \text{Player } A \begin{array}{c} 1 \\ 2 \end{array} & \left[\begin{array}{ccc} 6 & 8 & 6 \\ 4 & 12 & 2 \end{array}\right] \end{array}$$

[Ans. (I, I) ; (II, III); $v = 6$]

2.

$$\begin{array}{cc} & B \\ & \begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{IV} \end{array} \\ A \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{cccc} -5 & 3 & 1 & 20 \\ 5 & 5 & 4 & 6 \\ -4 & 2 & 0 & -5 \end{array}\right] \end{array}$$

[Raj. Univ. (M. Phil) 90]
[Ans. (2, III), $v = 4$]

3.

$$\left[\begin{array}{cccc} 8 & 15 & -4 & -2 \\ 19 & 15 & 17 & 16 \\ 0 & 20 & 15 & 5 \end{array}\right]$$

[Delhi. (OR.) 95]

4.

$$\begin{array}{cc} & B \\ A & \left[\begin{array}{ccc} 2 & 3 & 1/2 \\ 3/2 & 2 & 0 \\ 1/2 & 1 & 1 \end{array}\right] \end{array}$$

5.

$$\begin{array}{cc} & B \\ A & \left[\begin{array}{ccc} 1 & 8 & 4 \\ 6 & 4 & 5 \\ 0 & 1 & 2 \end{array}\right] \end{array}$$

[Ans. (1/4, 0, 3/4) for both players, $v = 7/8$]

6. Use dominance principle to reduce the following games to 2 × 2 games and hence solve them.

(i) $\begin{pmatrix} 2 & 0 & 3 \\ 3 & -1 & 1 \\ 5 & 2 & -1 \end{pmatrix}$ *(ii)* $\begin{pmatrix} 1 & -1 & 0 \\ -6 & 3 & -2 \\ 8 & -5 & 2 \end{pmatrix}$ *(iii)* $\begin{pmatrix} 8 & 5 & 8 \\ 8 & 6 & 5 \\ 7 & 4 & 5 \\ 6 & 5 & 6 \end{pmatrix}$ *(iv)* $\begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \end{array} \left[\begin{array}{ccc} 3 & -2 & 4 \\ -1 & 4 & 2 \\ 2 & 2 & 6 \end{array}\right]$

[Delhi B.Sc. (Math.) 91]

[Ans. ***(i)*** (1/2, 0, 1/2), (2/3, 1/3, 0), $v = 0$, ***(ii)*** (0, 7/12, 5/12), (0, 1/3, 2/3), $v = -1/3$.
(iii) (1/4, 1/4, 0, 0), (0, 3/4, 1/4), $v = 23/4$, ***(iv)*** (0, 0, 1), (2/5 , 3/5, 0), $v = 2$].

7.

$$\begin{array}{cc} & \begin{array}{ccccc} B_1 & B_2 & B_3 & B_4 & B_5 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \left[\begin{array}{ccccc} 4 & 4 & 2 & -4 & -6 \\ 8 & 6 & 8 & -4 & 0 \\ 10 & 2 & 4 & 10 & 12 \end{array}\right] \end{array}$$

[Ans. A (0, 4/9, 5/9), B (0, 7/9, 0, 2/9, 0) = $v = 34/9$]

8. Following is the payoff matrix for player A :

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \left[\begin{array}{ccccc} 2 & 4 & 3 & 8 & 4 \\ 5 & 6 & 3 & 7 & 8 \\ 6 & 7 & 9 & 8 & 7 \\ 4 & 2 & 8 & 4 & 3 \end{array}\right] \end{array}$$

Using dominance properties, obtain the optimum strategies for bbth the players and determine the value of the game.
[Ans. (III, I), $v = 6$ for A]
Note. Also, without using dominance property saddle point (6) exist in this problem.

9. A and B play a game in which each has three coins, a 5 p., a 10 p., and a 20 p. Each selects a coin without the knowledge of the others choice. If the sum of the coins is an odd amount, A wins B's coin; if the sum is even B wins A's coin. Find the best strategy for each player and the value of game. **[Kanpur 2000, 97]**

[**Hint.** Formulation of the game is as follows :

$$\begin{array}{cc} & B \\ & \begin{array}{ccc} 5P & 10P & 20P \end{array} \\ A\ \begin{array}{c} 5P \\ 10P \\ 20P \end{array} & \begin{bmatrix} -5 & 10 & 20 \\ 5 & -10 & -10 \\ 5 & -20 & -20 \end{bmatrix} \end{array}$$

[**Hint.** Reduce this game to 2×2 using dominance. III row is dominated by II, then III column in dominated by II].
[**Ans.** (1/2, 1/2, 0), (2/3, 1/3, 0), $v = 0$].

10. In a small town there are two discount stores *ABC* and *XYZ*. They are the only stores handle sundry goods. The total number of customers is equally divided between the two, because the price and quality of goods sold are equal. Both stores have good reputations in the community, and they render equally good customer services. Assume that a gain of customers by *ABC* is a loss to XYZ, and *vice-versa*.

		Strategies of *XYZ*		
		Press	Radio	Television
Strategies of *ABC*	Press	30	40	−80
	Radio	0	15	−20
	Television	90	20	50

Both stores plan to run annual pre-diwali sales during the first week of October. Sales are advertised through the local newspaper, radio and television media. With the aid of an advertising firm, *ABC* store constructed the same matrix given below (Figures in the matrix represent a gain or loss of customer). Find the optimal strategies for both stores and the value of the game.
[**Hint.** Use dominance to reduce the size to 2×2 game.
Ans. $\begin{pmatrix} \text{Press} & \text{Radio} & \text{T.V.} \\ 1/5 & 0 & 4/5 \end{pmatrix}, \begin{pmatrix} \text{Press} & \text{Radio} & \text{T.V.} \\ 0 & 13/15 & 2/15 \end{pmatrix}, v = 24$]

11. **(Media Problem).** Consider the following payoff matrix for two firms. What is the best mixed strategy for both the firms and also find out the value of the game ?

		Firm II		
		No advertising	Medium advertising	Large advertising
	No advertising	60	50	40
Firm I	Medium adversiging	70	70	50
	Large advertising	80	60	75

[Delhi (M.Com.) 90 Type]

[**Hint.** Use dominance to reduce it to 2×2 form, then solve].
[**Ans.** (0, 3/7 , 4/7), (0, 5/7, 2/7), $v = 150/7$]

12. Explain the terms 'Saddle point' and 'Dominance' in connection with the theory of games. Show that if dominance occurs in the payoff matrix of 2×2 game, then there is a saddle point. Is the converse true ? Solve the game whose payoff matrix is as given below.

$$\begin{bmatrix} 1 & 7 & 2 \\ 0 & 2 & 7 \\ 5 & 2 & 6 \end{bmatrix}$$

[**Hint.** Use dominance to reduce it to 2×2 form, then solve]. **[Raj. (M. Phil.) 93. 92]**

13. Given the payoff matrix for player *A*, obtain the optimum strategies for both the players and determine the value of the game. **[Jammu (M.B.A.) Jan. 99]**

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} 6 & -3 & 7 \\ -3 & 0 & 4 \end{bmatrix} \end{array}$$

[**Ans.** Optimum strategies for players *A* and *B* will be as follows :
$S_A = [\,1/4,\ 3/4\,]$ and $S_B = [\,1/4,\ 3/4,\ 0\,]$
Expected value of game $= \frac{3}{4}$.]

14. *A* is paid Rs. 8·00 if two coins turn both heads and Re. 1·00 if two coins turn both tails. *B* is paid Rs. 3·00 when the two coins do not match. Given the choice of being *A* or *B*, which one which one would you choose and what would be your strategy ? **[Delhi (M.B.A.) March 99]**
[**Ans.** Mixed strategies are $A\left(\frac{4}{5}, \frac{11}{15}\right)$; $B\left(\frac{4}{5}, \frac{1}{15}\right)$ and the expected value of the game is :
$v = \{8p - 3(1-p)\}\, q + \{-3p + (1-p)\}(1-q) = \left\{8 \times \frac{4}{15} - 3 \times \frac{11}{15}\right\} \times \frac{4}{15} + \left\{-3 \times \frac{4}{15} + \frac{11}{15}\right\} \times \frac{11}{15} = -\frac{1}{15}$]

15. Even though there are several manufacturers of scooters, two firms with branch names *Janata* and *Praja*, control their market in Western India. If both manufacturers make model changes of the same type for this market segment in the same year, their respective market shares remain constant. Likewise, if neither makes model changes, then also their market shares remain constant. The pay-off matrix in terms of increased decreased percentage market share under different possible conditions is given below :

		Praja		
		No change	Minor change	Major change
	No change	0	−4	−10
Janata	Minor change	3	0	5
	Major change	8	1	0

(i) Find the value of the game.

(ii) What change should Janata consider if this information is available only to itself? **[Rajasthan (M. Com.) 98]**

[**Ans.** The optimum strategies for the two players are :

$$S_A = [\,0\,,\ 1/6\,,\ 5/6\,] \text{ and } S_B = [\,0\,,\ 5/6\,,\ 1/6\,]$$

(ii) *Janata* may consider to have minor change with probability 5/6 and of major change with probability 1/6.]

16. In a small town, there ar only two stores that handle sundry goods—*ABC* and *XYZ*. The total number of customers is equally divided between the reputation in the community, and they render equally good customer service. Assume that a gain of customers by *ABC* is a loss to *XYZ* and vice versa. Both stores plan to run annual pre-diwali sales during the first week of November. Sales are advertised through a *local newspaper, radio* and *television* media. With the aid of an advertising firm store *ABC* constrcted the game matrix given below. (Figures in the matrix represent a gain or loss of customers).

		Strategy of XYZ		
		Newspaper	Radio	Television
Strategy of ABC	Newspaper	30	40	− 80
	Radio	0	15	− 20
	Television	90	20	50

Determine optimum strategies and the worth of such strategies for both *ABC* and *XYZ*.

[A.I.M.A. (P.G. Dip in Management), Dec. 96]

17. Two breakfast food manufacturers, *ABC* and *XYZ* are competing for an increased market share. The pay-off matrix, shown in the following table, shows the increase in market share for *ABC* and decrease in market share of *XYZ*.

ABC	XYZ: Give coupons	Decrease price	Maintain present strategy	Increase advertising
Give coupons	2	− 2	4	1
Decrease price	6	1	12	3
Maintain present strategy	− 3	2	0	6
Increase advertising	2	− 3	7	1

Simplyfy the problem by the rule of dominance and find optimum strategies for both the manufacturers and value of the game. **[Delhi (M.B.A.) Dec. 95]**

[**Ans.** The optimum strategies for both the manufacturers are that manufacturer *ABC* should adopt strategy 'decrease price' 50% time and strategy maintain present strategy 50% time. Similarly, manufacturer *XYZ* should adopt strategy 'give coupons 10% times and strategy 'decrease price' 90% times. The value of the game will be in favour of manufacturer *ABC* and increase in market share would be 3·5.]

18. Two firms are competing for business under the conditions so that one firm's gain is another firm's loss. Firm A's pay-off matrix is given below :

		Firm B: No advertising	Medium advertising	Heavy advertising
Firm A	No advertising	10	5	− 2
	medium advertising	13	12	15
	Heavy advertising	16	14	10

Suggest optimum strategies for the two firms and the net outcome thereof. **[Delhi (M. Com.) 94]**

19. A steel company is negotiating with its union for revision of wages to its employees. The management, with the help of a mediator, has prepared a pay-off matrix shown below. Plus sign represents wages increase, while negative sign stands for wage decrease. Union has also constructed a table which is comparable to that developed by management. The management does not have the specific knowledge of game theory to select the best strategy or (strategies) for the firm. You have to assist the management on the problem. What game value and strategies are available to the opposing group ?

Additional ocst to Settl Co. (Rs.)
Union strategies

		U_1	U_2	U_3	U_4
Steel Co. strategies	C_1	+ 2·50	+ 2·70	+ 3·50	− 0·20
	C_2	+ 2·00	+ 1·60	+ 0·80	+ 0·80
	C_3	+ 1·40	+ 1·20	+ 1·50	+ 1·30
	C_4	+ 3·00	+ 1·40	+ 1·90	0

[Delhi (M.B.A.) Nov. 98]

[**Hint.** Since the company represents the 'minimizing player' and the union of the 'maximizing player', the given pay-off matrix is recast as follows by interchanging rows and columns.

		Company strategies			
		C_1	C_2	C_3	C_4
	U_1	2·50	2·00	1·40	3·00
Union strategies	U_2	2·70	1·60	1·20	1·40
	U_3	3·50	0·80	1·50	1·90
	U_4	– 0·20	0·80	1·30	0

[**Ans.** Optimum strategy for the company is $\left(0, \frac{1}{13}, \frac{12}{13}, 0\right)$; for the union it is $\left(\frac{7}{13}, 0, \frac{6}{13}, 0\right)$ and the game value is $\frac{188}{1300}$ (representing increased wages).

20. Consider the game

		B 1	2	3
A	1	5	50	50
	2	1	1	0.1
	3	10	1	10

Verify that the strategies (1/6, 0, 5/6) for player A and (49/54, 5/54, 0) for player B are optimal and find the value of the game. **[JNTU (B. Tech.) 2003]**

19.15. GRAPHICAL METHOD FOR (2 × *n*) AND (*m* × 2) GAMES

The optimal strategies for a $(2 \times n)$ or $(m \times 2)$ matrix game can be located easily by a simple *graphical method.* This method enables us to reduce the $2 \times n$ or $m \times 2$ matrix game to 2×2 game that could be easily solved by the earlier methods.

If the graphical method is used for a particular problem, then the same reasoning can be used to solve any game with mixed strategies that has only two undominated pure strategies for one of the players.

Optimal strategies for both the players assign non-zero probabilities to the same number of pure strategies. It is clear that if one player has only two strategies, the other will also use two strategies. Hence, graphical method can be used to find two strategies of the player. The method can be applied to $3 \times n$ or $m \times 3$ games also by carefully drawing three dimensional diagram.

19·15-1. Graphical Method for 2 × *n* Games

Consider the $(2 \times n)$ game, assuming that the game does not have a saddle point.

Since the player A has two strategies, it follows that $x_2 = 1 - x_1$, $x_1 \geq 0$, $x_2 \geq 0$. Thus, for each of the pure strategies available to the player B, the expected payoff for the player A, would be as follows :

Table 19·27

			B y_1 B_1	y_2 B_2	y_3 B_3	...	y_n B_n
A	x_1	A_1	v_{11}	v_{12}	v_{13}	...	v_{1n}
	$1 - x_1$	A_2	v_{21}	v_{22}	v_{23}	...	v_{2n}

This shows that the player A's expected payoff varies linearly with x_1.

Table 19·28

B's Pure Strategies	A's Expected Payoff $E_j(x_1)$
B_1	$v_{11}x_1 + v_{21}(1 - x_1) = (v_{11} - v_{21})x_1 + v_{21}$
B_2	$v_{12}x_1 + v_{22}(1 - x_1) = (v_{12} - v_{22})x_1 + v_{22}$
:	: : :
B_n	$v_{1n}x_1 + v_{2n}(1 - x_1) = (v_{1n} - v_{2n})x_1 + v_{2n}$

According to the maximin criterion for mixed strategy games; the player A should select the value of x_1 so as to maximize his minimum expected payoff. This may be done by plotting the following straight lines :

$$E_1(x_1) = (v_{11} - v_{21})x_1 + v_{21}$$
$$E_2(x_1) = (v_{12} - v_{22})x_1 + v_{22}$$
$$\vdots \qquad \vdots \quad \vdots$$
$$E_n(x_1) = (v_{1n} - v_{2n})x_1 + v_{2n}$$

as functions of x_1. The lowest boundary of these lines will give the minimum expected payoff as function of x_1. The highest point on this lowest boundary would then give the *maximin* expected payoff and the optimum value of x_1 $(= x_1{}^*)$.

Now determine only two strategies for player B corresponding to those two lines which pass through the *maximin* point P (Fig. 19·1). This way, it is possible to reduce the game to 2×2 which can be easily solved either by using formulae given in **Sec. 19·13** or by *arithmetic method.*

Outlines of Graphical Method :

To determine maximin value $\underline{v}$, we take different values of x_1 on the horizontal line and values of $E(x_1)$ on the vertical axis. Since $0 \le x_1 \le 1$, the straight line $E_j(x_1)$ must pass through the points $\{0, E_j(0)\}$ and $\{1, E_j(1)\}$, where $E_j(0) = v_{2j}$ and $E_j(1) = v_{1j}$. Thus the lines $E_j(x_1) = (v_{1j} - v_{2j})\, x_1 + v_{2j}$ for $j = 1, 2, \ldots, n$ can be drawn as follows :

Step 1. Construct two vertical axes, *axis* 1 at the point $x_1 = 0$ and *axis* 2 at the point $x_1 = 1$.

Step 2. Represent the payoffs $v_{2j}, j = 1, 2, \ldots, n$ on *axis* 1 and payoff $v_{1j}, j = 1, 2, \ldots, n$ on *axis* 2.

Step 3. Join the point representing v_{1j} on *Axis* 2 to the point representing v_{2j} on *axis* 1. The resulting straightline is the expected payoff line $E_j(x_1), j = 1, 2, \ldots, n$.

Step 4. Mark the lowest boundary of the lines $E_j(x_1)$ so plotted, by *thickline* segments. The *highest point* on this *lowest boundary* gives the *maximin* point P and identifies the two critical moves of player B.

If there are more than two lines passing through the maximin point P, there are ties for the optimum mixed strategies for player B. Thus any two such lines with *opposite* sign slopes will define an alternative optimum for B.

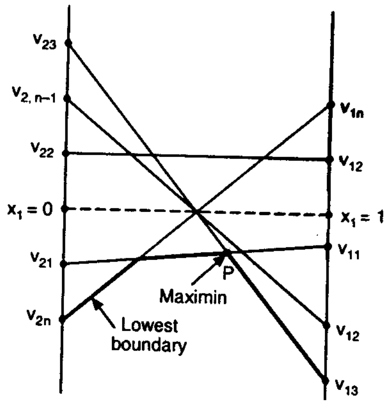


Fig. 19.1 Graphical solution of 2xn games.

19·15-2. Graphical Solution of $m \times 2$ Games

The $(m \times 2)$ games are also treated in the like manner except that the *minimax* point P is the *lowest point* on the *uppermost boundary* instead of highest point on the lowest boundary.

From this discussion, it is concluded that any $(2 \times n)$ or $(m \times 2)$ game is basically equivalent to a (2×2) game.

Now each point of the discussion is explained by solving numerical examples for $(2 \times n)$ and $(m \times 2)$ games.

Q. Explain the graphical method of solving $(2 \times n)$ and $(m \times 2)$ games.

Example 13. *Solve the following* (2×3) *game graphically.*

Table 19·29

			y_1 I	y_2 II	y_3 III
A	x_1	I	1	3	11
	$1 - x_1$	II	8	5	2

[JNTU (Mech. & Prod.) 2004, 03, 02; Agra 99; Delhi B.Sc. (Math) 91]

Solution. This game does not have a saddle point. Thus the player A's expected payoff corresponding to the player B's pure strategies are given (*Table 19·30*).

Three expected payoff lines are :

$$E(x_1) = -7x_1 + 8, \; E(x_1) = -2x_1 + 5 \text{ and } E(x_1) = 9x_1 + 2$$

and can be plotted on a graph as follows **[see Fig. 19·2]**

Table 19·30

B's Pure Strategies	A's Expected Payoff $E(X_1)$
I	$E(x_1) = 1.x_1 + 8(1 - x_1) = -7x_1 + 8$
II	$E(x_1) = 3.x_1 + 5.(1 - x_1) = -2x_1 + 5$
III	$E(x_1) = 11x_1 + 2(1 - x_1) = 9x_1 + 2$

First, draw two parallel lines one unit apart and mark a scale on each. These two lines will represent two strategies available to the player A. Then draw lines to represent each of player B's strategies.

For example, to represent the player B's Ist strategy, join mark 1 on scale I to mark 8 on scale II; to represent the player B's second strategy, join mark 3 on scale I to mark 5 on scale II, and so on. Since the expected payoff $E(x_1)$ is the function of x_1 alone, these three expected payoff lines can be drawn by taking x_1 as x-axis and $E(x_1)$ as y- axis.

Points A, P, B, C on the lowest boundary (shown by a thick line in Fig. 19·2) represent the lowest possible expected gain to the player A for any value of x_1 between 0 and 1. According to the *maximin* criterion, the player A chooses the best of these worst outcomes.

Clearly, the ***highest*** point P on the ***lowest*** boundary will give the largest expected gain PN to A. So best strategies for the player B are those which pass through the point P. Thus, the game is reduced to 2 × 2 (*Table 19·31*).

Now, by solving the simultaneous equations

$3x_1 + 5x_2 = v,\ 11x_1 + 2x_2 = v,\ x_1 + x_2 = 1$ (*For player A*)

$3y_2 + 11y_3 = v,\ 5y_2 + 2y_3 = v,\ y_2 + y_3 = 1$ (*For player B*)

the solution of the game is obtained as follows :

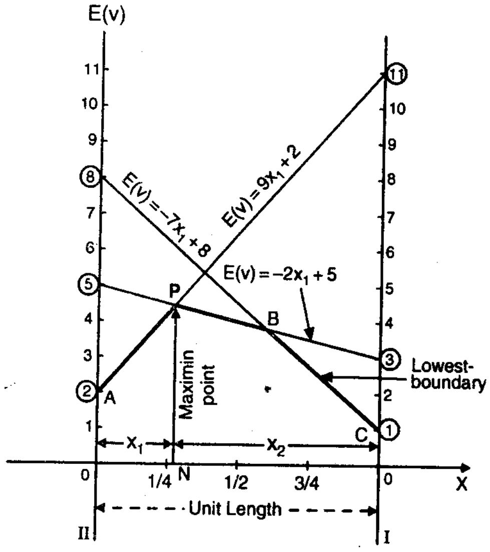


Fig. 19.2 Graphical representation for solving (2 × *n*) game.

(*i*) The player A chooses the optimal mixed strategy $(x_1, x_2) = (3/11, 8/11)$,

(*ii*) The player B chooses the optimal mixed strategy $(y_1, y_2, y_3) = (0, 2/11, 9/11)$,

(*iii*) The value of the game to the player A is $v = 49/11$.

Table 19·31

		B: II	B: III
A	I	3	11
	II	5	2

Example 14. *Solve the game graphically whose payoff matrix for the player A is given in Table 19·32 :*

Table 19·32

		B: I	B: II
A	I	2	4
	II	2	3
	III	3	2
	IV	−2	6

Solution. The game does not have a saddle point. Let y_1 and y_2 $(= 1 - y_1)$ be mixed strategies of the player B.

The four straight lines thus obtained are :

$$E(y_1) = -2y_1 + 4,\ E(y_1) = -y_1 + 3,$$

$$E(y_1) = y_1 + 2,\ E(y_1) = -8y_1 + 6,$$

and these are plotted in Fig. 19·3. In this case, the *minimax* point is determined as the *lowest point P* on the uppermost boundary. Lines intersecting at the minimax point P correspond to the player A's pure strategies I and III. This indicates $x_2 = x_4 = 0$. Thus, the reduced game is given in *Table 19·34*.

Table 19·33

A's Pure Strategies	*B*'s Expected Payoff $E(y_1)$
I	$E(y_1) = 2y_1 + 4(1 - y_1)$
II	$E(y_1) = 2y_1 + 3(1 - y_1)$
III	$E(y_1) = 3y_1 + 2(1 - y_1)$
IV	$E(y_1) = -2y_1 + 6(1 - y_1)$

Now, solve this (2 × 2) game by solving the simultaneous equations :

$2x_1 + 3x_3 = v,\ 4x_1 + 2x_3 = v,\ x_1 + x_3 = 1$ (*For A*)

$2y_1 + 4y_2 = v,\ 3y_1 + 2y_2 = v,\ y_1 + y_2 = 1$ (*For B*)

to get the solution :

Table 19·34

		B: I	B: II
A	I	2	4
	III	3	2

(*i*) The player A chooses the optimal mixed strategy, $(x_1, x_2, x_3, x_4) = (1/3, 0, 2/3, 0)$.

(*ii*) The player B chooses the optimal mixed strategy, $(y_1, y_2) = (2/3, 1/3)$.

(*iii*) The value of the game to the player A is $v = 8/3$.

Remark. If there are more than two lines passing through the maximin (minimax) point P, this would imply that there are many ties for optimal mixed strategies for the player B. Thus, any two lines having opposite signs for their slops will define an alternative optimum solutions.

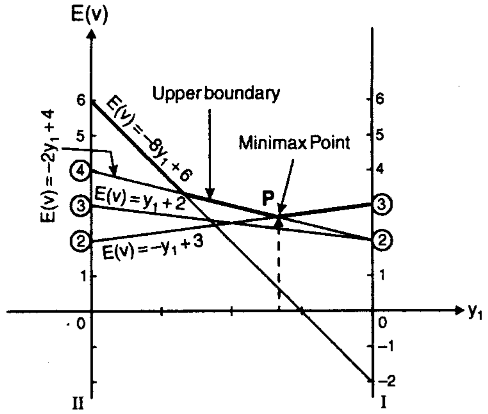


Fig. 19.3. Graphical representation for ($m \times 2$) game.

19·15-3. Illustrative Examples

Example 15. *Solve the following* (2×4) *game.*

		B I	II	III	IV
A	I	2	2	3	− 1
	II	4	3	2	6

[Agra M.Sc. (Maths.) 99; Meerut M.Sc. (Math.) 99, 96]

Solution. This game does not have saddle point. Thus, the player A's expected payoffs corresponding to the player B's pure strategies are given below (*Table 19·35*) :

Table 19·35

B's Pure Strategies	A's Expected Payoff $E(x_1)$
I	$E(x_1) = -2x_1 + 4$
II	$E(x_1) = -x_1 + 3$
III	$E(x_1) = x_1 + 2$
IV	$E(x_1) = -7x_1 + 6$

These four straight lines are then plotted in Fig. 19·4.

It follows from *Fig.* 19·4 that maximin occurs at $x_1 = 1/2$. This is the point of intersection of *any* two of the lines joining (2) to (3); (3) to (2); (6) to (− 1). As mentioned in the above remark, any *two lines having opposite signs for their slopes will define an alternative* optimum solution. The combination of lines $E(x_1) = -x_1 + 3$ and $E(x_1) = -7x_1 + 6$ must be excluded as being non- optimal. So the game can be reduced to (2×2) in the following manner.

It is also important to note that the average of above two payoff matrices (*Table 19·36* and *19·37*) will also be the additional possibility of reducing the game to (2×2). Thus, the additional possibility of (2×2) game will also yield a new optimal solution which mixes three strategies II, III and IV. Then (2×2) game is solved by solving the governing simultaneous equations.

Table 19·36
1st possibility

		B II	III
A	I	2	3
	II	3	2

Table 19·37
2nd possibility

		B III	IV
A	I	3	− 1
	II	2	6

Table 19·38
Additional possibility

		B	
A	I	$\frac{2+3}{2} = 5/2$	$\frac{3-1}{2} = 1$
	II	$\frac{3+2}{2} = 5/2$	$\frac{2+6}{2} = 4$

The first possibility of the solution of (2×4) game with reduced (2×2) matrix is :

(*i*) The player A chooses the optimal mixed strategy $(x_1, x_2) = (1/2, 1/2)$.

(*ii*) The player B chooses the optimal mixed strategy $(y_1, y_2, y_3, y_4) = (0, 1/2, 1/2, 0)$.

(*iii*) The value of the game to the player A is $2\frac{1}{2}$.

Similarly, solution of the game with reduced matrix in 2nd possibility is :

(*i*) The player A chooses the optimal mixed strategy $(1/2, 1/2)$.

(*ii*) The player B chooses the optimal mixed strategy $(0, 0, 7/8, 1/8)$.

(*iii*) The value of the game to the player A is $2\frac{1}{2}$.

Solution of the game with reduced matrix in additional possibility can be obtained easily, because it has a saddle point $5/2$. So, the value of the game to the player A is $5/2$. It has been observed that the formulae in *Sec.* ***19·13*** will yield an incorrect solution in this case.

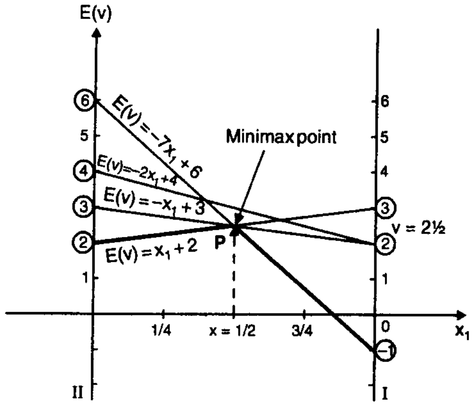


Fig. 19.4

Example 16. *Two firms A and B make colour and black & white television sets. Firm A can make either 150 colour sets in a week or an equal number of black & white sets, and make a profit of Rs. 400 per colour set and Rs. 300 per black & white set. Firm B can, on the other hand, make either 300 colour sets, or 150 colour and 150 black & white sets, or 300 black & white sets per week. It also has the same profit margin on the two sets as A. Each week there is a market of 150 colour sets and 300 black & white sets and the manufacturers would share market in the proportion in which they manufacture a partiuclar type of set.*

Write the pay-off matrix of A per week. Obtain graphically A's and B's optimum strategies and value of the game. **[Bombay (M.M.S.) 97]**

Solution. For firm A, the strategies are :

A_1 : make 150 colour sets,

A_2 : make 150 black & white sets.

For firm B, the strategies are :

B_1 : make 300 colour sets,

B_2 : make 150 colour and 150 black & white sets,

B_3 : make 300 black and white sets.

For the combination A_1B_1, the profit to firm A would be : $\dfrac{150}{150+300} \times 150 \times 400 = \text{Rs. } 20{,}000$

wherein $(150/150 + 300)$ represents share of market for A, 150 is the total market for colour television sets and 400 is the profit per set. In a similar way, other profit figures may be obtained as shown in the following pay-off matrix :

$$\begin{array}{cc} & \text{B's strategy} \\ & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \text{A's strategy} \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 20{,}000 & 30{,}000 & 60{,}000 \\ 45{,}000 & 45{,}000 & 30{,}000 \end{bmatrix} \end{array}$$

Since no saddle point exists, we shall determine optimum mixed strategy. The data are plotted on graph as shown in the adjoining Fig. 19·5 :

Lines joining the pay-offs on axis I with the pay-offs on axis II represents each of B's strategies. Since firm A wishes to maximize his minimum expected pay-off, we consider the highest point P on the lower envelope of A's expected pay-off equation. This point P represents the maximin expected value of the game for firm A. The lines B_1 and B_3 passing through P, define the relevant moves B_1 and B_3 that alone from B needs to adopt. The solution to the original 2×3 game, therefore, reduces to that of the simple game with 2×2 pay-off matrix as follows :

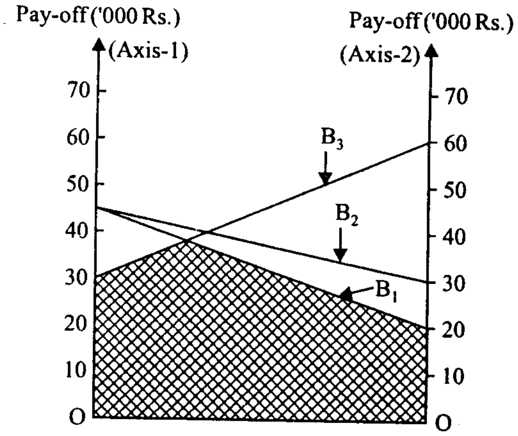


Fig. 19.5 : Graphic solution to the Game

$$\begin{array}{ccc} & & B\text{'s strategy} \\ & & \begin{array}{cc} B_1 & B_2 \end{array} \\ A\text{'s strategy} & \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 20{,}000 & 60{,}000 \\ 45{,}000 & 30{,}000 \end{bmatrix} \end{array}$$

Correspondingly,

$$p_1 = \frac{a_{22}+a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{30{,}000-45{,}000}{(20{,}000+30{,}000)-(60{,}000+45{,}000)} = \frac{3}{11}$$

$$q_1 = \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{30{,}000-60{,}000}{(20{,}000+30{,}000)-(60{,}000+45{,}000)} = \frac{6}{11}$$

$$v = \frac{a_{11}a_{22}-a_{12}a_{21}}{(a_{11}+a_{21})-(a_{12}+a_{21})} = \frac{20{,}000\times 30{,}000-60{,}000\times 45{,}000}{(20{,}000+30{,}000)-(60{,}000+45{,}000)} \simeq 38{,}182$$

Example 17. *Solve the following game by graphical method.*

		B: y_1	y_2	y_3	y_4
A	x_1	19	6	7	5
	x_2	7	3	14	6
	x_3	12	8	18	4
	x_4	8	7	13	−1

[VTU (BE Mech.) 2002]

Solution.

Step 1. The first step is to search for a saddle point. There is no saddle point in this problem.

Step 2. The second step is to observe if the game can be reduced by dominance. Since all the cell entries in column 2 are less than the corresponding values in columns 1 and 3, hence columns 1 and 3 are dominated by column 2 and thus the reduced matrix becomes as shown on the right.

		y_2	y_4
A	x_1	6	5
	x_2	3	6
	x_3	8	4
	x_4	7	−1

Again, since all the cell entries for row 3 are more than those for row 4, hence row 3 dominates row 4 and the matrix is thus reduced to the following (3 × 2) game.

		B: y_2	$y_4 = 1 - y_2$
A	x_1	6	5
	x_2	3	6
	x_4	8	4

Step 3. Now the above (3 × 2) matrix game can be solved by graphical method proceeding as in *Example 14.*

We can, thus, immediately reduce the (3 × 2) game to the following (2 × 2) game which can be easily solved by arithmetic method (since it has no saddle point).

		B: II	IV		
A	I	6	5	3	3/4
	II	3	6	1	1/4
		1	3		
		1/4	3/4		

Therefore, optimal strategies are :

(*i*) A (3/4, 1/4, 0, 0), (*ii*) B (0, 1/4, 0, 3/4) (*iii*) Value of the game, $v = \frac{6 \times 1 + 3 \times 6}{1 + 3} = 5\frac{1}{4}$

19·15-4 Method of Subgames for (2 × *n*) or (*m* × 2) Games

To explain the method we consider the following interesting example.

Example 18. *Two airlines operate the same air-route, both trying to get as large a market as possible. Based on a certain market, daily gains and losses in rupees are shown in table below, in which positive values favour airline A and negative values favour airline B. Find the solution for the game.*

			Airline B		
			Does nothing	Advertises special rates	Advertises special features (*i.e.* movies, fine food)
			1	2	3
Airline A	Advertises special rates	1	275	−50	−75
	Advertises special features (*i.e.* movies, fine food)	2	125	130	150

Solution.

***Step.* 1.** First observe that the game neither has a saddle point, nor it can be reduced by dominance. This game can be solved by algebraic method as described in [**Sec. 19·17**], but we shall solve this game here by the *method of subgames.*

***Step.* 2.** This (2 × 3) game can be considered as three (2 × 2) subgames :

Subgame I

A \ B	1	2
1	275	−50
2	125	130

(Deleting col. 3)

Subgame II

A \ B	1	3
1	275	−75
2	125	150

(Deleting col. 2)

Subgame III

A \ B	2	3
1	−50	−75
2	130	150

(Deleting col. 1)

***Step* 3.** It is observed that airline *B* which has more number of columns (than the number of rows for *A*), has more flexibility, generally resulting in a better strategy. In order to find optimum strategy for airline *B*, all the above three (2 × 2) subgames must be solved. We solve them below by *Arithmetic Method.*

Subgame 1. (*No saddle point*) :

(*i*) The strategy for *A* is, (1/66, 65/66)

(*ii*) The strategy for *B* is, (36/66, 30/66, 0)

(*iii*) Value of the game = Rs. $\frac{275 \times 1 + 125 \times 65}{66}$ = Rs. 127·30.

A \ B	1	2			
1	275	−50	5	1	1/66
2	125	130	325	65	65/66
	180	150			
or	6	6			
or	36	30			
or	36/66	30/66			

Subgame 2. (*No saddle point*) :

(*i*) The strategy for *A* is : (1/15 , 14/15)

(*ii*) The strategy for *B* is : (9/15 , 0, 6/15)

(*iii*) The value of the game, *v* = Rs. $\frac{275 \times 3 - 75 \times 2}{5}$ = Rs. 135.

A \ B	1	3			
1	275	−75	25	1	1/15
2	125	150	350	14	14/15
	225	150			
	3	2 ,			
	3/5	2/5			
	9/15	6/15			

Subgame 3. (*Has a saddle point*) :

Thus, this game has a saddle point (2, 2). So the solution is *A* (0, 1), *B* (0, 1, 0), *v* = Rs. 130.

Now since airline *B* has the flexibility to play any two out of the available courses of action, it will play those strategies for which the loss occurring to the airline is minimum. Since all the values for the subgames are positive, the airline *A* is the winner. Hence airline *B* will play subgame 1 for which the loss is minimum, *i.e.* Rs. 127·30. Hence the complete solution to the problem is :

A \ B	2	3	Row Min.
1	−50	−75	−75
2	130*	150	130*
Col. Max.	130*	150	

Optimum strategies : *A* (1/66 , 65/66), *B* (36/66 , 30/66, 0)

Value of the game, *v* = 127·30

Note Carefully. Here subgame 3 has a saddle point, hence arithmetic method should not be applied to solve it. If it is applied, the resulting solution will be incorrect.

EXAMINATION PROBLEMS

Use graphical method to reduce the following games and hence solve :

1.

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 3 & -3 & 4 \\ -1 & 1 & -3 \end{bmatrix} \end{array}$$

[Ans. $A\,(4/11, 7/11)$, $B\,(0, 7/11, 4/11)$, $v = -5/11$]

2.

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & 2 \end{bmatrix} \end{array}$$

[Ans. (3/7, 4/7), (2/7, 0, 5/7), $v = 8/7$]

3.

$$\begin{array}{cc} & \textbf{Player } B \\ \textbf{Player } A & \begin{bmatrix} 2 & 7 \\ 3 & 5 \\ 11 & 2 \end{bmatrix} \end{array}$$

[Ans. (9/14, 0, 5/14), (5/14, 9/14), $v = 73/14$]

4.

$$\begin{array}{cc} & \textbf{Player } B \\ \textbf{Player } A & \begin{bmatrix} 1 & 2 \\ 5 & 6 \\ -7 & -9 \\ -4 & -3 \\ 2 & 1 \end{bmatrix} \end{array}$$

[Meerut M.Sc (Math.) 98 BP, 96.]

[Hint. Upper boundary is the line ($5 \to 6$)]

[Ans. (0, 1, 0, 0); (1, 0), $v = 5$]

5. Solve the game whose payoff matrix is given below.

(i)

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 6 & 2 & 7 \\ 1 & 9 & 3 \end{bmatrix} \end{array}$$

[Meerut 99]

(ii)

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & -3 \end{bmatrix} \end{array}$$

(iii)

$$\begin{bmatrix} -4 & 3 & -1 \\ 6 & -4 & -2 \end{bmatrix}$$

6. Solve the following 2×4 game graphically :

$$\begin{array}{ccc} & & \textbf{Player } B \\ & & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \textbf{Player } A & \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 2 & 1 & 0 & -2 \\ 1 & 0 & 3 & 3 \end{bmatrix} \end{array}$$

[Ans. (2/5, 3/5), (0, 4/5, 0, 1/5), $v = 2/5$] **[Agra 92; Madurai B.Sc. (Comp. Sc.) 92, Jadavpur M.Sc. (Math) 92]**

7. Obtain the optimal strategies for both persons and the value of the game for zero-sum two-person game whose payoff matrix is given as follows :

$$\begin{array}{cc} & \textbf{Player } A \\ \textbf{Player } B & \begin{bmatrix} 1 & 3 & -1 & 4 & 2 & -5 \\ -3 & 5 & 6 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

[Ans. (0, 3/5, 0, 2/5, 0, 0), (4/5, 1/5), $v = 17/5$]

8. Two companies A and B are competing for the same product. Their different strategies are given in the following payoff matrix :

(i)

$$\begin{array}{ccc} & & \textbf{Company } A \\ & & \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \textbf{Company } B & \begin{array}{c} B_1 \\ B_2 \end{array} & \begin{bmatrix} 2 & -2 & 3 \\ -3 & 5 & -1 \end{bmatrix} \end{array}$$

(ii)

$$\begin{array}{cc} & \textbf{Company } B \\ \textbf{Company } A & \begin{bmatrix} 4 & -3 & 3 \\ -3 & 1 & -1 \end{bmatrix} \end{array}$$

What are the best strategies for both the companies ? **[Hint.** Here III column is dominated by 1/2 (I + II) column.]
Find out the value of the game.

[Ans. (0, 4/11, 7/11), (6/11, 5/11), $v = 13/11$] **[Ans.** $A\,(4/11, 7/11)$, $B\,(4/11, 7/11, 0)$, $v = -5/11$]

9. Solve the following zero-sum games where player B plays with player A.

(i)

$$\begin{array}{cc} & \textbf{Player } B \\ \textbf{Player } A & \begin{bmatrix} 2 & -4 & 6 & -3 & 5 \\ -3 & 4 & -4 & 1 & 0 \end{bmatrix} \end{array}$$

[JNTU (B. Tech.) 2003]

(ii)

$$\begin{array}{cc} & \textbf{Player } B \\ \textbf{Player } A & \begin{bmatrix} -6 & 7 \\ 4 & -5 \\ -1 & -2 \\ -2 & 5 \\ 7 & 6 \end{bmatrix} \end{array}$$

[VTU 2002; Agra 99; IAS (Main) 98]

[Ans. (i) $A\,(4/9, 5/9)$, $B\,(4/9, 0, 5/9, 0)$, $v = -7/9$. (ii) $A\,(0, 0, 0, 13/20, 7/20)$, $B\,(11/20, 9/20)$, $v = 23/20$]

10. Obtain the optimal strategies for both persons and the value of the game for two person zero-sum game whose payoff matrix is as follows :

$$\begin{array}{ccc} & & \textbf{Player } B \\ & & \begin{array}{cc} B_1 & B_2 \end{array} \\ \textbf{Player } A & \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{array} & \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 6 \\ 4 & 1 \\ 2 & 2 \\ -5 & 0 \end{bmatrix} \end{array}$$

[Ans. (0, 3/5, 0, 2/5, 0, 0), (4/5, 1/5), $v = 17/5$]

From above two examples we observe that the matrix method described above is highly systematic but involves laborious computations in the case of large payoff matrices. For example, the number of square submatrices of a matrix of higher order is itself a very large number and so it becomes exceedingly tiresome to find the adjoint of a square matric of higher order. Thus in order to remove this difficulty a short-cut matrix method for $n \times n$ games is developed.

19·16-1. A Short-Cut Matrix Method for $n \times n$ Games

A short-cut matrix method is described below to solve any $n \times n$ games quite efficiently, though it provides only *one* optimal solution.

***Step* 1.** Let $\mathbf{A} = \{v_{ij}\}$ be the $n \times n$ payoff matrix. Obtain a new matrix **C** whose first column is obtained from **A** by subtracting its 2nd column from 1st; second column is obtained by subtracting **A**'*s* 3rd column from 2nd, and so on till the last column of **A** has been taken into consideration. Thus **C** will be an $n \times (n-1)$ matrix. Likewise, obtain a new matrix **R** of order $(n-1) \times n$ by subtracting its successive row's from the preceding ones as is done for columns to obtain **C** as above.

***Step* 2.** Augment the matrix **A** as below :

$$\begin{array}{cc|c} \boxed{\;\mathbf{A}\;} & \mathbf{C} & \\ \mathbf{R} & & \\ \hline & & \end{array}$$

***Step* 3.** Compute the magnitude of oddments corresponding to each row and each column of **A**.
The oddment corresponding to ith row of **A** is defined as the determinant $|\mathbf{C}_i|$, where $\mathbf{C}_i$ is obtained from **C** by deleting its ith row.
Similarly, oddment corresponding to jth column of $\mathbf{A} = |\mathbf{R}_j|$, where $\mathbf{R}_j$ is obtained from **R** by deleting its jth column.

***Step* 4.** Write down the **magnitude** of oddments (ignoring the –ve sign, if any) against their respective rows and columns as shown is *step 2.*

***Step* 5.** Now check whether the sum of row oddments is equal to the sum of column oddments.
(i) If so, the oddments expressed as the fractions of the grand total will provide the optimal strategies.
(ii) If not, the method fails.

Q. **1.** Briefly explain the matrix method and short-cut matrix method for solving a rectangular game.

2. Show that if the elements of a payoff matrix are all integers, then the value of the game is a rational number.

3. Show that the game with payoff matrix $\begin{pmatrix} a & 0 & 1 \\ 1 & a & 0 \\ 0 & 1 & a \end{pmatrix}$ has a unique solution. Determine the value of the game.

4. Show that the game whose payoff matrix is $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ where $a > b > c > 0$, has a unique solution. What is the value of the game ? What can be said about the solution if $a > b > c$ and $c < 0$?

The short-cut metrix method can be easily understood by the following numerical example.

Example 21. *Solve the following game by using short-cut matrix method.*

		B			
		I	II	III	Row min
	I	7	1	7	1
A	II	9	–1	1	–1
	III	5	7	6	(5) ←maximin value
Column max.		9	(7)	(7)	

minimax value

Solution.

***Step* 1.** The first step is to search for a saddle point. There is no saddle point in this problem, but the value of the game lies between 5 and 7 (*i.e.* between maximin and minimax values).

***Step* 2.** The next step is to observe if the given matrix can be reduced by dominance. We find that the given matrix cannot be reduced. So we now solve this matrix game by *method of matrices* as explained in the following steps.

***Step* 3.** Subtract each row from the row above it (*i.e.* subtract 2nd row from 1st and 3rd row from the 2nd) and write down the values below the matrix. Similarly, subtract each column from the column to its left (*i.e.* subtract 2nd column from the 1st and 3rd column from the 2nd) and write down the results to the right of the matrix. Thus, we get the following table.

		B				
		I	II	III		
A	I	7	1	7	6	−6
	II	9	−1	1	10	−2
	III	5	7	6	−2	1
		−2	2	6		
		4	−8	−5		

***Step* 4.** Now, compute the oddments for A_1 , A_2 , A_3 and B_1 , B_2 , B_3 .

Oddment for $A_1 = \det \begin{vmatrix} 10 & -2 \\ -2 & 1 \end{vmatrix} = 10 - 4 = 6,$

" " $A_2 = \det \begin{vmatrix} 6 & -6 \\ -2 & 1 \end{vmatrix} = 12 - 6 = 6,$

" " $A_3 = \det \begin{vmatrix} 6 & -6 \\ 10 & -2 \end{vmatrix} = -12 + 60 = 48$

Oddment for $B_1 = \det \begin{vmatrix} 2 & 6 \\ -8 & -5 \end{vmatrix} = -10 + 48 = 38,$

" " $B_2 = \det \begin{vmatrix} -2 & 6 \\ 4 & -5 \end{vmatrix} = 24 - 10 = 14,$

" " $B_3 = \det \begin{vmatrix} -2 & 2 \\ 4 & -8 \end{vmatrix} = 16 - 8 = 8.$

***Step* 5.** Next, put down these oddments (as shown in the table below) neglecting their signs. Since both the sums of oddments are same (60 each), this is a solution to the game. If the sums are different, both players do not use all of their courses of actions in their strategies, and this method fails.

	I	II	III				
I	7	1	7	6	1	1/10	3/30
II	9	−1	1	6	1	1/10	3/30
III	5	7	6	48	8	8/10	24/30
	38	14	8	60			
	19	7	4				
	19/30	7/30	4/30				

Thus the optimum strategies are :

(i) *A* (3/30, 3/30, 24/30)

(ii) *B* (19/30, 7/30, 4/30)

(iii) Value of the game, $v = \frac{7 \times 1 + 9 \times 1 + 5 \times 8}{1 + 1 + 8} = 5\frac{3}{5}$.

Remark. The short-cut method of matrices can be applied only when the sum of *vertical oddments* is equal to the sum of *horizontal oddments.* In other words, if both the players use all their plays in their best strategies. The method breaks down when the players do not use all their courses of action in their best strategies. In such a case the method of linear programming may be used.

EXAMINATION PROBLEMS

Use matrix method to solve the games whose payoff matrices are given below (apply short-cut method wherever applicable) :

1. (a) $\begin{bmatrix} 2 & -2 & 3 \\ -2 & 5 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

[Ans. (2/3, 1/3), (7/12, 0) , $v = 1/3$] **[Ans.** (1/2, 1/2), (0, 1, 0), $v = 1$] **[Ans.** (0, 1/2, 1/2), (0, 1/2, 1/2), $v = 0$]

3. Use matrix method to solve the game whose payoff matrix is

$$\begin{bmatrix} 2 & 3 & 11 \\ 7 & 5 & 2 \end{bmatrix}$$

[Ans. (9/11, 2/11), (0, 3/11, 8/11), $v = 49/11$]

4. Let the payoff matrix of a rectangular game be $\begin{bmatrix} \lambda & \lambda & \lambda \\ \lambda & 3 & 4 \\ \lambda & 5 & 1 \end{bmatrix}$. Show that for all values of λ, the value of the game is λ. For what value of λ does the row player (column player) have an infinite number of optimal strategies.

19.17. ALGEBRAIC METHOD FOR THE SOLUTION OF A GENERAL GAME

The algebraic method is a direct attempt to solve unknowns from the relationships (equations 19·21a) in *Sec.* **19·10** for player *A*, and similarly for player *B*. Although this method becomes quite lengthy when there are more strategies (courses of action) for players *A* and *B*. Such large games can be solved first by transforming the problem into a linear programming problem and then solving it by the *simplex method* on an electronic computer.

First, suppose that all inequalities given in (19·21a) hold as equations. Then solve these equations for unknowns. Sometimes equations are not consistent. In such cases, one or more of the inequalities must hold as strict inequalities (with '>' or '<' signs). Hence, there will be no alternative except to rely on trial-and-error method for solving such games. Following important theorems will be helpful in making the computations easier.

Theorem 19·7. *If for any j ($j = 1, 2, 3, \dots, n$) $v_{1j}x_1 + v_{2j}x_2 + \dots + v_{mj}\, x_m > v$, then $y_j = 0$, and similarly, if for any i ($i = 1, 2, 3, \dots, m$) $v_{i1}y_1 + v_{i2}y_2 + \dots + v_{in}y_n < v$, then $x_i = 0$.*

Alternative Statement : *Let v be the value of an $m \times n$ game. If for an optimum strategy $\mathbf{x}^* \in S_m$, $E(\mathbf{x}^*, \mathbf{e_j}) > v$ for some $\mathbf{e_j} \in S_n$, then every strategy $\mathbf{y}^* \in S_n$ has $y_j^* = 0$,*

Similarly, if every optimal strategy $\mathbf{y}^* \in S_n$, then every optimal strategy $\mathbf{x}^* \in S_m$ has $x_i^* = 0$.

Proof. We know that for any optimal strategy $\mathbf{x}^* \in S_m$, we always have

$$E(\mathbf{x}^*, \mathbf{e_j}) \geq v \text{ for all } \mathbf{e}_j \in S_n \qquad \dots(1)$$

We are given that $E(\mathbf{x}^*, \mathbf{e_j}) > v$ for some $\mathbf{e_j} \in S_n$. If possible let us suppose $y_j^* \neq 0$ (*i.e.* $y_j^* > 0$).

Then $\quad y_j^*\, E(\mathbf{x}^*, \mathbf{e_j}) > y_j^*\, v. \qquad \dots(2)$

Now $\quad E(\mathbf{x}^*, \mathbf{y}^*) = \Sigma\, y_k^*\, E(\mathbf{x}^*, \mathbf{e_k}) \quad = \sum_{k \neq j} y_k^*\, E(\mathbf{x}^*, \mathbf{e_k}) + y_j^*\, E(\mathbf{x}^*, \mathbf{e_j})$

or $\quad v > \sum_{k \neq j} y_k^*\, v + y_j^*\, v$ or $v > (\sum_{k \neq j} y_k^* + y_j^*)v$ or $v > \Sigma\, y_k^*\, v$ [from (1) and (2)]

or $\quad v > v$ (since $\Sigma\, y_k^* = 1$) which is a contradiction.

Hence our assumption is wrong; and therefore, we must have $y_j^* = 0$.

Similarly, the second part of the theorem can be proved.

Theorem 19.8. *Let v be the value of an $m \times n$ game. If $\mathbf{y}^* \in S_n$ is an optimal strategy for the column vector with $y_j^* \neq 0$, then every optimal strategy $\mathbf{x}^* \in S_m$ for the row player must satisfy $E(\mathbf{x}^*, \mathbf{e_j}) = v$* for all $\mathbf{e_j} \in S_n$.

Similarly, if the optimal strategy $\mathbf{x}^* \in S_m$ has $x_i^* \neq 0$, then every optimal strategy $\mathbf{y}^* \in S_n$ must satisfy $E(\mathbf{e_i}, \mathbf{y}^*) = v$ for all $\mathbf{e}_i \in S_m$.

Proof. Left as an exercise to the students.

Theorem 19.9. *If the player A's optimal policy is a mixed strategy in which exactly **r** pure strategies have non-zero probabilities, then the player B's optimal strategy also uses **r** pure strategies.*

Proof. Left as an exercise to the students.

Q. 1. If $X = \{x_i\}$ is an optimal mixed strategy for *A* and $Y = \{y_j\}$ is optimal mixed strategy for *B*, in a rectangular game specified by an $m \times n$ matrix $\{v_{ij}\}$ and v is the value of the game, then prove that if $E(e_i, Y) < v$, then $x_i = 0$, where $\mathbf{e}_i$ is the ith unit vector in m-dimensions and $E(\mathbf{e}_i, \mathbf{Y})$ is the expected amount received by *A* when he uses the strategy $\mathbf{e}_i$ and *B* uses the strategy *Y*.

2. Let *E* be the expectation function of an $m \times n$ rectangular game whose value is v and let

$Y^* = \| x_1^*, x_2^*, \dots, x_m^* \|$ and $Y^* = \| y_1^*, y_2^*, \dots, y_n^* \|$

be any optimal strategies for P_1 and P_2, respectively. Then,

(a) for any i such that $E(i, Y^*) < v$, show that $x_1^* = 0$ (b) for any j such that $v < E(x^*, j)$ show that $y_j^* = 0$

[Delhi B.Sc. (Math.) 93]

19·17-1. Illustrative Examples

Example 22. *Find the value and optimal strategies for two players of the rectangular game whose payoff matrix is given by*

			B		
			y_1 I	y_2 II	y_3 III
	x_1	I	1	−1	−1
A	x_2	II	−1	−1	3
	x_3	III	−1	2	−1

Solution. First, it is seen that this game does not have a saddle point. Also, this game cannot be reduced to (2×2) by the property of dominance. Hence, this game can be solved by the algebraic method.

Step 1. Let (x_1, x_2, x_3) and (y_1, y_2, y_3) denote the optimal mixed strategies for players A and B, respectively, and v be the vlaue of the game to the player A.

Now, for the player A, following relationships (*as obtained in Sec. 19·10*) can be established:

$$1x_1 + (-1)\,x_2 + (-1)\,x_3 \ge v$$
$$-1\,x_1 + (-1)\,x_2 + 2\,x_3 \ge v$$
$$-1\,x_1 + 3x_2 + (-1)\,x_3 \ge v.$$

Similarly, for the player B,

$$1.y_1 + (-1)\,y_2 + (-1)\,y_3 \le v$$
$$-1.y_1 + (-1)\,y_2 + 3\,y_3 \le v$$
$$-1.y_1 + 2y_2 + (-1)\,y_3 \le v.$$

Additional relationship required to ensure that x_1, x_2, x_3 and y_1, y_2, y_3 are probabilities, are:

$$x_1 + x_2 + x_3 = 1,\ x_1, x_2, x_3 \ge 0 \text{ and } y_1 + y_2 + y_3 = 1,\ y_1, y_2, y_3 \ge 0.$$

Now the values of seven unknowns x_1, x_2, x_3 ; y_1, y_2, y_3 and v satisfying above relationships are to be evaluated.

Step 2. Suppose all inequalities hold as equations, then

$$x_1 - x_2 - x_3 = v \quad \text{...(i)} \qquad y_1 - y_2 - y_3 = v \quad \text{...(v)}$$
$$-x_1 - x_2 + 2x_3 = v \quad \text{...(ii)} \qquad -y_1 - y_2 + 3y_3 = v \quad \text{...(vi)}$$
$$-x_1 + 3x_2 - x_3 = v \quad \text{...(iii)} \qquad -y_1 + 2y_2 - y_3 = v \quad \text{...(vii)}$$
$$x_1 + x_2 + x_3 = 1 \quad \text{...(iv)} \qquad y_1 + y_2 + y_3 = 1. \quad \text{...(viii)}$$

Now with the help of the equation (iv), equations (i), (ii) and (iii) give us

$$x_1 = \frac{v+1}{2},\ x_2 = \frac{v+1}{4},\ x_3 = \frac{v+1}{3}$$

and substituting these values of x_1, x_2, x_3 in equation (iv), we get

$$\frac{v+1}{2} + \frac{v+1}{4} + \frac{v+1}{3} = 1.$$

Therefore, $\quad v + 1 = 12/13$ or $v = -1/13$.

Hence $\quad x_1 = 6/13,\ x_2 = 3/13,\ x_3 = 4/13$.

Again with the help of equation (viii), equations (v), (vi) and (vii) give us

$$y_1 = \frac{v+1}{2},\ y_2 = \frac{v+1}{3},\ y_3 = \frac{v+1}{4}$$

and substituting these values of y_1, y_2, y_3 in equation (viii),

$$\frac{v+1}{2} + \frac{v+1}{4} + \frac{v+1}{3} = 1.$$

Therefore, $\quad v + 1 = 12/13$ or $v = -1/13$, which was expected also by minimax theorem.

Thus, $\quad y_1 = 6/13,\ y_2 = 4/13$ and $y_3 = 3/13$.

Hence, the solution of the game is:

(*i*) Optimal mixed strategy for the player A is $(6/13, 3/13, 4/13)$

(*ii*) Optimal mixed strategy for the player B is $(6/13, 4/13, 3/13)$

(*iii*) The value of the game to the player A is $= -1/13$.

Example 23. *In the following* 3×3 *game, find optimal strategies and the value of the game.* **[Delhi (OR.) 92]**

Table 19·39

		B		
		I	II	III
	I	3	−2	4
A	II	−1	4	2
	III	2	2	6

Solution. It can be observed that this game does not have a saddle point. The size of the matrix of this game can be further reduced by using the dominance property as follows:

Step 1. From the player B's point of view, every element of IIIrd column is greater than the corresponding element in the Ist column. Hence the column III is dominated by the Ist column. So the size of the game can be reduced by deleting III column.

Step 2. Further, none of the rows or columns dominates the other. But, the average of I and II rows in the reduced matrix (a), *i.e.*

$$\frac{3+(-1)}{2}=1, \frac{-2+4}{2}=1,$$

is less than the corresponding elements of the III row. Hence the III row dominates the average of I and II rows. Thus, delete either I or II row. If we delete the I row, the 2×2 reduced game (b) is obtained.

(a)

		B: I	B: II
A	I	3	−2
	II	−1	4
	III	2	2

(b)

		B: I	B: II
A	II	−1	4
	III	2*	2

(c)

		B: I	B: II
A	I	3	−2
	III	2	2*

Step 3. Now, the cell (III, I) is the saddle point. For element '2' marked '*' is minimum in its row and maximum in its column.

Thus, the solution of the game is given by

(i) Optimal strategy for the player A is pure strategy (0, 0, 1),

(ii) Optimal strategy for the player B is also pure strategy (1, 0, 0),

(iii) The value of the game to the player A is 2.

Remark. If IInd row is deleted, (instead of Ist in the reduced 3×2 matrix) the saddle point is obtained in the cell (III, II), as shown in matrix (c) above. Consequently, the optimal strategy for the player A becomes (0, 0, 1) and for the player B becomes (0, 1, 0). The value of the game will not change.

Example 24. *Solve the game with the following payoff matrix by algebraic method.*

Table 19.40

		B: I	B: II
A	I	3	2
	II	4	1

Solution. The game has a saddle point (2) which immediately gives the required solution, but it will be solved by algebraic method to make the procedure clear.

Let (x_1, x_2) and (y_1, y_2) be the optimal strategies for player A and B, respectively. Therefore, relationships existing for optimality are :

$$3x_1+4x_2 \ge v \quad ...(i) \qquad 3y_1+2y_2 \le v \quad ...(iv)$$

$$2x_1+1x_2 \le v \quad ...(ii) \qquad 4y_1+y_2 \le v \quad ...(v)$$

$$x_1+x_2=1 \quad ...(iii) \qquad y_1+y_2=1 \quad ...(vi)$$

and
$$x_1, x_2, y_1, y_2 \ge 0. \quad ...(vii)$$

First, suppose all inequalities hold as equations, *i.e.*

$$3x_1+4x_2=v, \quad 2x_1+x_2=v, \quad x_1+x_2=1$$

$$3y_1+2y_2=v, \quad 4y_1+y_2=v, \quad y_1+y_2=1.$$

Eliminate x_2 and y_2 from equations (i), (ii), (iv) and (v) with the help of equations (iii) and (vi), to obtain

$$-x_1+4=v, \quad x_1+1=v, \quad y_1+2=v, \quad 3y_1+1=v,$$

or
$$x_1=3/2, \; y_1=1/2.$$

But x_1 can never be greater than unity (by the definition of probability). Thus, equations are not consistent. Hence one or more of the inequalities must hold as strict inequalities.

Now, use trial and error method together with the *Theorems* 19.4 & 1.5. Let two of the equations be strict inequalities :

$$-x_1+4>v \quad \text{(which implies } y_1=0)$$

$$x_1+1=v$$

$$y_1+2=v$$

$$3y_1+1<v \quad \text{(which implies } x_2=0).$$

Since $y_1=0$, $v=2$. Now, put $v=2$ in $x_1+1=v$ to get the value $x_1=1$.

Hence the solution is given by :

(*i*) Optimal strategy of the player A is $(x_1, x_2) = (1, 0)$

(*ii*) Optimal strategy of the player B is $(y_1, y_2) = (0, 1)$,

(*iii*) The value of the game to the player A is 2.

Remark. In view of the Theorem 1.9 **(Sec. 19·17)**, the total number of equations that remain to solve (excluding the strict inequalities) at each trial should always be even.

EXAMINATION PROBLEMS

Solve the following games by algebraic method.

1. $\begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix}$ **2.** $\begin{bmatrix} -1 & -2 & 8 \\ 7 & 5 & -1 \\ 6 & 0 & 12 \end{bmatrix}$ **[JNTU 97]** **3.** $\begin{bmatrix} -2 & -4 \\ -1 & 3 \\ 1 & 1 \end{bmatrix}$ **[JNTU (B. Tech.) 2003]**

[Ans. (2/9, 5/9, 2/9), $v = 1/9$].

4. $\begin{bmatrix} -1 & 2 & 1 \\ 1 & -2 & 2 \\ 3 & 4 & -3 \end{bmatrix}$ **[Meerut 90]** **5.** $\begin{bmatrix} 3 & -1 & 1 & 2 \\ -2 & 3 & 2 & 3 \\ 2 & -2 & -1 & 1 \end{bmatrix}$

[Ans. (7/23, 6/23, 10/23) ; (17/46, 20/46, 9/46), $v = 15/23$]

6. $A = \begin{bmatrix} 3 & -3 & 4 \\ -1 & 1 & -3 \end{bmatrix}$ **7.** $\begin{bmatrix} 2 & 3 & -1 \\ 4 & -1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

[Ans. (4/11, 7/11), (0, 7/11, 4/11), $v = -5/11$] **[Ans.** (8/30, 5/30, 17/30), (13/30, 11/30, 6/30), $v = 53/30$]

8. Consider the game with the following payoff matrix. Verify that the optimal strategy for either player is to mix his 3 pure strategies equally, what is the value of this game.

$$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

[Ans. (1/3, 1/3, 1/3) for both players, $v = 0$]

9. Solve the following zero-sum game for two persons. Obtain the best strategies for both players and the value of the game :

$$\begin{array}{cc} & \textbf{Player } B \\ & \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \textbf{Player } A \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \end{array} & \begin{bmatrix} 1 & -1 & 3 \\ 2 & -1 & 2 \\ -1 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix} \end{array}$$

[Ans. A (0, 1/4, 3/4, 0), B (1/4, 3/4, 0), $v = -1/4$]

10. Use algebraic method to solve the games whose payoff matrices are :

(a) $\begin{bmatrix} 1 & -1 & 0 \\ -6 & 3 & -2 \\ 8 & -5 & 2 \end{bmatrix}$ **(b)** $\begin{bmatrix} 3 & 4 & -2 \\ -3 & 0 & 1 \\ -1 & -4 & 2 \end{bmatrix}$ **(c)** $\begin{bmatrix} 5 & 2 & -3 \\ 4 & 2 & 7 \\ -4 & 5 & -2 \end{bmatrix}$ **(d)** $\begin{bmatrix} 3 & -1 & -3 \\ -3 & 3 & -1 \\ -4 & -3 & 3 \end{bmatrix}$

[Ans. (a) (0, 7/12, 5/12), (0, 1/3, 2/3), $v = -1/3$. (b) (21/52, 12/52, 19/52), (2/13, 3/13, 8/13), $v = 2/13$. (c) (39/118, 54/118, 25/118), (30/118, 3/118, 85/118). (d) (4/9, 11/45, 14/45), (14/45, 11/45, 4/9) $v = -29/45$]

11. Solve the game whose payoff matrix is given by

$$\begin{bmatrix} 1 & 7 & 2 \\ 0 & 2 & 7 \\ 5 & 1 & 6 \end{bmatrix}$$

[JNTU (B. Tech.) 98]

[Ans. (2/5, 0, 3/5), (3/5, 2/5, 0), $v = 17/5$]

12. Use concept of dominance to reduced the size of the matrix of the given problem to 2×3 matrix and solve the game. The payoff matrix of the game is

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} 1 & 8 & 3 \\ 6 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix} \end{array}$$

[Ans. (1/6, 5/6, 0), (0, 1/3, 2/3), $v = 14/3$]

19.18. AN ITERATIVE METHOD FOR APPROXIMATE SOLUTION

The algebraic method is generally adopted to solve the game for which the graphical method cannot be applied, but the games with large payoff matrices are extremely tiresome to solve by algebraic method. For such large games the iterative method is also very powerful to hand as well as machine computations. By this method, the approximate value of the game can be evaluated upto any desired degree of accuracy. Optimal strategies can also be determined, but not so satisfactorily.

While adopting this method, it is assumed that each player acts under the assumption that past is the best guide to the future and will play in such a manner so as to maximize his expected gain (or to minimize his expected loss).

This method can best be explained by the following example.

		B		
		I	II	III
	I	1	−1	−1
A	II	−1	−1	3
	III	−1	2	−1

Example 25. *Find the value and optimal strategies for two players of the rectangular game whose payoff matrix (for the player A) is given below :* **[Meerut 2002; Delhi (OR.) 93; Jodhpur M.Sc. (Math) 92]**

Solution. In this method, the player A arbitrarily selects any row and places it under the matrix. Here, select I row arbitrarily. The player B examines this row and chooses a column corresponding to the *smallest* number in the row. This is column III. Column III is then placed to the right of the matrix. The player A examines this column and chooses a row corresponding to the *largest* number in this column. This is row II. Row II is then added to the first row last chosen and the sum of the two rows is placed beneath the row last chosen. The player B chooses a column corresponding to the smallest number in the new row and adds this column to the column last chosen. In the case of a tie (equality of numbers which prevents either player from being victorious) that row or column must be chosen by a coin flipping process. But, in this example, in case of a tie, the player will select the row or column different from his last choice. This procedure may be continued in the like manner. Ten iterations are shown with smallest elements in each succeeding row and largest elements in each succeeding column encircled. Approximate strategies after *ten* iterations are found by dividing the number of encircled elements in each row or column by total number of iterations. Thus the player A's approximate strategy is (5/10, 2/10, 3/10), and the player B's approximate strategy is (6/10, 2/10, 2/10), (*Table 19·41*)

Table 19·41

		B													
		I	II	III											
	I	1	−1	−1	−1	−2	−3	−2	−1	$\boxed{0}$	$\boxed{1}$	$\boxed{2}$	$\boxed{3}$	$\boxed{2}$	5/10
A	II	−1	−1	3	$\boxed{3}$	$\boxed{2}$	1	0	−1	−2	−3	−4	−5	−2	2/10
	III	−1	2	−1	−1	1	$\boxed{3}$	$\boxed{2}$	$\boxed{1}$	0	−1	−2	−3	−4	3/10
		1	−1*	−1											
		0	−2	2											
		−1	−3	5											
		−2	−1	4											
		−3	1	3											
		−4	3	2											
		−3	2	1											
		−2	1	0											
		−1	0	−1*											
		0	−1	−2											
		6/10	2/10	2/10											

The upper bound for the value of the game can be determined by dividing the largest element, 2, in the last column by the total number of iterations, 10. Likewise, the lower bound can be determined by dividing the smallest element, − 2, in the last row by the number of iterations, 10.

Thus, $-2/10 < v < 2/10$ or $-1/5 < v < 1/5$.

The approximate solution thus obtained is given by :

(*i*) Optimal strategy for player A is (5/10, 2/10, 3/10)

(*ii*) Optimal strategy for player B is (6/10, 2/10, 2/10)

(*iii*) The value of the game lies between 1/5 and − 1/5.

Other approximate solutions can also be obtained with more iterations.

Q. 1. Explain the iterative method of getting an approximate solution to a game problem.
2. Discuss whether the sequence of approximate optimal strategies for the row (column) player converges.
3. Show that for a 2×2 game with a saddle point, the iterative method leads to the true value of the game.

EXAMINATION PROBLEMS

1. Obtain an approximate solution by *Iterative method*. If possible, find the exact solution also.

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 1 & 0 & 2 \\ 3 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix} \end{array}.$$

[**Ans.** (1/4, 1/4, 1/2), (1/3, 1/3, 1/3), $v = 1$]

2. Solve the following game approximately :

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 2 & 3 & -1 & 0 \\ 5 & 4 & 2 & -2 \\ 1 & 3 & 8 & 2 \end{bmatrix} \end{array}$$

[**Ans.** (0, 1/10, 9/10), (4/10, 0, 0, 6/10), $14/10 \le v \le 16/10$]

3. Obtain the approximate solution by iterative method of the games given below. If possible, find the exact solution also :

(a) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 3 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 & 6 \\ 2 & 1 & 3 \\ 6 & 2 & 1 \end{bmatrix}$

[**Ans.** (a) (4/10, 3/10, 3/10), (5/10, 2/10, 3/10), $8/10 < v < 12/10$.
(b) (5/10, 0, 5/10), (2/10, 7/10, 1/10), $25/10 < v < 29/10$]

4. Find to an accuracy of two places of decimals, the value of the games whose payoff matrices are :

(a) $\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 2 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \\ 2 & 5 & 0 \end{bmatrix}$

[**Ans.** (a) (1/3, 1/3, 1/3), (1/3, 1/3, 1/3), $v = 0{\cdot}67$. (b) (5/8, 1/8, 2/8), (3/8, 3/8, 2/8), $13/8 < v < 15/8$]

5. Show by the iterative method, that the game whose payoff matrix is $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ has a value zero.

19.19. SUMMARY OF METHODS FOR RECTANGULAR GAMES

In order to solve the two-person zero-sum games, we must proceed in the systematic order as follows :

***Step* 1.** First of all search for a saddle point. If it is found, the problem is readily solved.

***Step* 2.** If no saddle point is found, then use the concept of dominance to reduce the size of the matrix game. If dominance is found, delete the dominated row(s) and/or column(s). Each matrix thus obtained must be further checked for dominance.

***Step* 3.** If the size of the reduced matrix become (2×2) with no saddle point, it can be solved by *arithmetic* and *algebraic* methods described in ***Sections* 19·13-1 & 19·17.**

***Step* 4.** If the size of the reduced matrix becomes $(2 \times n)$ or $(m \times 2)$, use graphical method to reduce it to (2×2) matrix and then solve it by arithmetic or algebraic method.
If graphical method is not to be used, the game can still be solved by algebraic method and method of subgames. All these methods are described in ***Sections* 19·15-4 & 19·17.**

***Step* 5.** If the reduced size of the matrix becomes (3×3) or higher, then *algebraic method, method of matrices, simplex method of linear programming,* or *the iterative method of approximate solution* can be used for solving it. These methods are discussed in ***Sections* 19·17, 19·16, 19·12 & 19·18.**

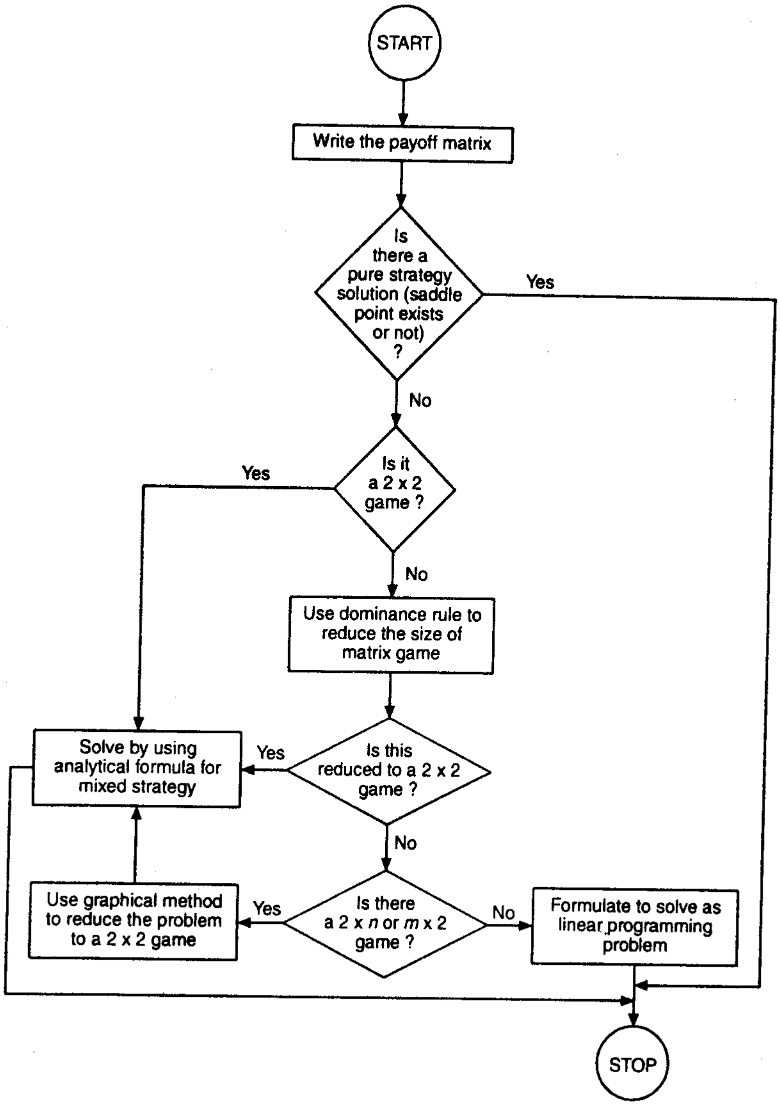
Q. 1. "Game theory provides a systematic quantitative approach for analysing competitive situations in which the competitions make use of logical processes and techniques in order to determine an optimal strategy for winning." Comment.
2. Summarize the systematic methods for solving the rectangular games.

19·20. BUSINESS APPLICATIONS : BIDDING PROBLEMS

Business problems often involve bidding for contracts, *e.g.*, getting the opportunity for service or bidding for the rights to get some privileges such as *contracts, land, property, licences* and *concessions*, etc. The types of bidding problems are : *(i)* Open or Auction bids, *(ii)* Closed bids.

Some simple bidding problems can be easily done by the techniques of game theory. Following examples will illustrate the procedure.

FLOW CHART FOR SYSTEMATIC APPLICATION



Example 26. (Open or Auction Bids). *Two items of worth Rs. 75 and Rs. 125 are to be auctioned at a public sale. There are only two bidders A and B. Bidder A has Rs. 100 available and bidder B has Rs. 130 available. What should be their strategies if each bidder is interested in maximizing his own return.*

[Meerut (Maths.) 98 BP]

Solution. Let the successive increase of bids be Rs. λ. At any bid, each bidder has an option to increase the bid or to leave the opponents bid stand. Further, let B has bid, Rs. x on the Ist item (of value Rs. 75). Then A will think as follows :

If bidder A permits B to win the Ist item for Rs. x, then B will have only Rs. $(130 - x)$ for bidding on the IInd item. Obviously, B cannot bid for the IInd item more than Rs. $(130 - x)$. So, A will surely win the IInd item in Rs. $(130 - x + \lambda)$. hence the A's gain allowing B to win the Ist item for Rs. x, will be

$$\text{Rs. } [125 - (130 - x + \lambda)] \quad \text{or} \quad \text{Rs. } [x - \lambda - 5].$$

Alternatively, if A bids for Rs. $(x + \lambda)$ for the Ist item and B permits him to win at this bid, the A's gain will be Rs. $[75 - (x + \lambda)]$ or Rs. $[75 - x - \lambda]$.

Since A wishes to maximize his return, he should bid $x + \lambda$ for the Ist item, subject to the condition

$$75 - x - \lambda \geq x - \lambda - 5 \quad \text{or} \quad x \leq 40.$$

Therefore, A should bid for the Ist item till $x \leq 40$. In case $x > 40$, he should allow B to win the Ist item for that bid.

Likewise, B's gains in two alternatives are : Rs. $[125 - (100 - y) - \lambda]$ and Rs. $[75 - (y + \lambda)]$ where y denotes the A's bid for the first item.

So B should bid Rs. $(y + \lambda)$ for the Ist item subject to the condition :

$$75 - (y + \lambda) \geq 125 - (100 - y) - \lambda \quad \text{or} \quad y \leq 25.$$

Clearly, A will then purchase the Ist item in Rs. 25, because he can increase his bid without any loss upto Rs. 40, and B will purchase the IInd item in Rs $(100 - 25) = 75$, because A (after winning the Ist in Rs. 25) cannot increase his bid for the IInd item more than Rs. 75. Therefore, B will purchase the second item in Rs. 75. So the gain to A is Rs. $(75 - 25) =$ Rs. 50 and to B is Rs. $(125 - 75) =$ Rs. 50.

Note. In this problem each bidder has an amount less than the total values of the two items. It is also assumed here that A knows the amount available with B.

Example 27. (Closed Bids). *Two objects of worth Rs. 80 and Rs. 100 are to be bid simultaneously by two bidders A and B. Both have intention of devoting a total sum of Rs. 100 to the two bids. If each uses a minimax criterion, find the resulting bids.*

Solution. In this problem, bids are closed because they are to be made simultaneously.

Suppose Rs. a_1 and Rs. a_2 are the A's optimum bids for the Ist and IInd object respectively. A's best bids are those which give the same amount of gain to A on both items. If p denotes the total profit associated with a successful bid, then

$$2p = (80 - a_1) + (100 - a_2) \quad \text{or} \quad 2p = 180 - (a_1 + a_2).$$

Since both have decided to spend only Rs. 110 for both the bids, $(a_1 + a_2) = 110$. Therefore,

$$2p = 180 - 110 \quad \text{or} \quad p = \text{Rs. } 35.$$

Hence $\quad a_1 = 80 - p = 80 - 35 = \text{Rs. } 45$ and $a_2 = 100 - p = 100 - 35 = \text{Rs. } 65.$

Thus the optimum bids for A are Rs. 45 and Rs. 65 for the Ist and IInd items, respectively.

Proceeding likewise, B's optimum bids can be determined. The optimum bids for B will be the same as that of A's optimum bids.

Q. 1. What is game theory ? Discuss its importance to business decisions.

2. What is game theory ? Include in your answer various approaches in solving strategies and game values.

3. Describe the role of 'theory of games' for scientific decision making.

4. "A game refers to a situation of business conflict". Discuss.

5. Describe some of the applications of game theory. What are its limitations ?

19·21. LIMITATIONS OF GAME THEORY

Game theory which was initially received in literature with great enthusiasm as holding promise, has been found to have a lot of limitations. The major limitations are summarised below :

1. The assumption that the players have the knowledge about their own payoffs and payoffs of others is rather unrealistic. He can only make a guess of his own and his rivals' strategies.
2. As the number of players increase in the game, the analysis of the gaming strategies become increaseingly complex and difficult. In practice, there are many firms in an oligopoly situation and game theory cannot be very helpful in such situations.
3. The assumptions of maximin and minimax show that the players are risk-averse and have complete knowledge of the strategies. These do not seem practical.
4. Rather than each player in an oligopoly situation working under uncertain conditions, the players will allow each other to share the secrets of business in order to work out a collusion, Then the mixed strategies are not very useful.

However, inspite of its limiations, game theory provides insight into the operations of oligopoly markets.

Q. Briefly explain the major limitations of game theory. **[JNTU (B. Tech.) 2003; Delhi (M.Com.) 90]**

19.22. SADDLE POINT FOR THE FUNCTION OF SEVERAL VARIABLES

Let $f(\mathbf{x}, \mathbf{y})$ be a real valued function of two vectors $\mathbf{x}$ and $\mathbf{y}$, $\mathbf{x} \in E^n$, $\mathbf{y} \in E^m$. If $f(\mathbf{x}, \mathbf{y})$ is such that $\mathbf{x}$ is fixed and $\mathbf{y}$ is varied then $f(\mathbf{x}, \mathbf{y})$ has a minimum for some value of $\mathbf{y}$. It is denoted by $\Phi = \min_y f(\mathbf{x}, \mathbf{y})$.

If $\mathbf{x}$ is given some other fixed value, it is possible to find another value of ϕ. Thus, for different values of $\mathbf{x}$, the values of ϕ can be obtained provided ϕ exists in every case. This implies that ϕ is a function of $\mathbf{x}$ represented by $\phi(\mathbf{x}) = \min_y f(\mathbf{x}, \mathbf{y})$.

Now, if $\phi(\mathbf{x})$ has a maximum for some value of $\mathbf{x}$, then it can be written as $\max_x \phi(\mathbf{x}) = \max_x \min_y f(\mathbf{x}, \mathbf{y})$.

Similar interpretation can be given for the expression $\min_y \max_x f(\mathbf{x}, \mathbf{y})$.

For this, first find a maximum of $f(\mathbf{x}, \mathbf{y})$ with respect to $\mathbf{x}$ keeping $\mathbf{y}$ fixed, and then find the minimum of the function thus obtained with respect to $\mathbf{y}$.

Definition. *A point* $(\mathbf{x}_0, \mathbf{y}_0)$, $\mathbf{x}_0 \in E^n$, $\mathbf{y}_0 \in E^m$ *is said to be a saddle point of* $f(\mathbf{x}, \mathbf{y})$ *if*

$$f(\mathbf{x}, \mathbf{y}_0) \le f(\mathbf{x}_0, \mathbf{y}_0) \le f(\mathbf{x}_0, \mathbf{y}). \qquad \text{....(19·50)}$$

Theorem 19.9. *Let* $f(\mathbf{x}, \mathbf{y})$ *be such that both* $\max_x \min_y f(\mathbf{x}, \mathbf{y})$ and $\min_y \max_x f(\mathbf{x}, \mathbf{y})$ *exist. Then*

$$\max_y \min_x f(\mathbf{x}, \mathbf{y}) \le \min_x \max_y f(\mathbf{x}, \mathbf{y}) \quad \text{[Delhi (OR.) 92]} \qquad \text{...(19·51)}$$

Proof. Choose two arbitrary points $\mathbf{x}_0$ and $\mathbf{y}_0$ in E^n and E^m respectively, then

$$\max_x f(\mathbf{x}, \mathbf{y}_0) \ge f(\mathbf{x}_0, \mathbf{y}_0) \quad \text{and} \quad \min_x f(\mathbf{x}_0, \mathbf{y}) \le f(\mathbf{x}_0, \mathbf{y}_0)$$

Hence,
$$\max_x f(\mathbf{x}, \mathbf{y}_0) \ge \min_y f(\mathbf{x}_0, \mathbf{y}). \qquad \text{...(19·52)}$$

Since $\mathbf{y}_0$ is arbitrarily chosen point in E^m, the inequality (1·52) must be true even if $\mathbf{y}_0$ be a point for which $\max_x f(\mathbf{x}, \mathbf{y})$ has the minimum value. Hence $\min_y [\max_x f(\mathbf{x}, \mathbf{y})] \ge \min_y f(\mathbf{x}_0, \mathbf{y})$.

Also, since $\mathbf{x}_0$ is any point in E^n, the inequality (19·52) will hold even if $\mathbf{x}_0$ is chosen to make $\min f(\mathbf{x}, \mathbf{y})$ maximum. So, $\min_y [\max_x f(\mathbf{x}, \mathbf{y})] \ge \max_x [\min_y f(\text{bol}x, y)]$.

Hence the theorem is proved.

Cor. 1. *Let* $\{v_{ij}\}$ *be an* $m \times n$ *matrix. Then* $\min_j [\max_i v_{ij}] \ge \max_i [\min_j v_{ij}]$ ...(19·53)

Proof. Consider the matrix $\{v_{ij}\}$ as a real valued function $f(i, j)$ of two variables i and j such that $1 \le i \le m$ and $1 \le j \le n$, then the equation (1·53) immediately follows from the equation (19·51).

Theorem 19.10. *Let* $f(\mathbf{x}, \mathbf{y})$ *be such that both* $\min_y \max_x f(\mathbf{x}, \mathbf{y})$ *and* $\max_x \min_y f(\mathbf{x}, \mathbf{y})$ *exist. Then necessary and sufficient condition for the existence of a saddle point* $(\mathbf{x}_0, \mathbf{y}_0)$ of $f(\mathbf{x}, \mathbf{y})$ is that

$$\min_y [\max_x f(\mathbf{x}, \mathbf{y})] = f(\mathbf{x}_0, \mathbf{y}_0) = \max_x [\min_y f(\mathbf{x}, \mathbf{y})].$$

Proof. The condition is necessary. Let $(\mathbf{x}_0, \mathbf{y}_0)$ be the saddle point so that inequality (19·50) holds.

Since $f(\mathbf{x}_0, \mathbf{y}_0) \ge f(\mathbf{x}, \mathbf{y}_0)$ for all $\mathbf{x} \in E^n$, $f(\mathbf{x}_0, \mathbf{y}_0) \ge \max_x f(\mathbf{x}, \mathbf{y}_0)$.

But $\max_x f(\mathbf{x}, \mathbf{y}_0) \ge \min_y [\max_x f(\mathbf{x}, \mathbf{y})]$. Therefore $f(\mathbf{x}_0, \mathbf{y}_0) \ge \min_y \max_x f(\mathbf{x}, \mathbf{y})$.

Again, from the inequality (19·50)

$$f(\mathbf{x}_0, \mathbf{y}_0) \le \min_y f(\mathbf{x}_0, \mathbf{y}) \qquad [\because f(\mathbf{x}_0, \mathbf{y}_0) \le f(\mathbf{x}_0, \mathbf{y})]$$

Also, $\max\limits_{x} [\min\limits_{y} f(\mathbf{x}, \mathbf{y})] \geq \min\limits_{y} f(\mathbf{x}_0, \mathbf{y}_0)$. Hence $\max\limits_{x} [\min\limits_{y} f(\mathbf{x}, \mathbf{y})] \geq f(\mathbf{x}_0, \mathbf{y}_0)$

Thus, $\max\limits_{x} \min\limits_{y} f(\mathbf{x}, \mathbf{y}) \geq f(\mathbf{x}_0, \mathbf{y}_0) \geq \min\limits_{y} \max\limits_{x} f(\mathbf{x}, \mathbf{y})$.

But, by virtue of (19·51), the only conclusion drawn is

$$\min_{y} \max_{x} f(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}_0, \mathbf{y}_0) = \max_{x} \min_{y} f(\mathbf{x}, \mathbf{y}). \qquad \text{...(19·54)}$$

The condition is sufficient. To prove this, assume that (1·54) is true. Let the minimum of $\max\limits_{x} f(\mathbf{x}, \mathbf{y})$ occur at $\mathbf{y}_0$ and the maximum of $\min\limits_{y} f(\mathbf{x}, \mathbf{y})$ occur at $\mathbf{x}_0$. Then by virtue of (19·54),

$$\max_{x} f(\mathbf{x}, \mathbf{y}_0) = \min_{y} f(\mathbf{x}_0, \mathbf{y}). \qquad \text{...(19·55)}$$

By definition of minima, $f(\mathbf{x}_0, \mathbf{y}_0) \geq \min\limits_{y} f(\mathbf{x}, \mathbf{y}_0)$

and thus (19·55) gives $f(\mathbf{x}_0, \mathbf{y}_0) \geq \max\limits_{x} f(\mathbf{x}, \mathbf{y}_0)$

which implies that $f(\mathbf{x}_0, \mathbf{y}_0) \geq f(\mathbf{x}, \mathbf{y}_0)$ for all $\mathbf{x}$.

Similarly, by definition of maxima, $f(\mathbf{x}_0, \mathbf{y}_0) \leq \max\limits_{x} f(\mathbf{x}, \mathbf{y}_0)$

and, therefore, again from (19·55) $f(\mathbf{x}_0, \mathbf{y}_0) \leq \min\limits_{y} f(\mathbf{x}_0, \mathbf{y})$

which implies that $f(\mathbf{x}_0, \mathbf{y}_0) \leq f(\mathbf{x}_0, \mathbf{y})$ for all $\mathbf{y}$.

Thus, $f(\mathbf{x}_0, \mathbf{y}) \geq f(\mathbf{x}_0, \mathbf{y}_0) \geq f(\mathbf{x}, \mathbf{y}_0)$.

Hence, by definition, $(\mathbf{x}_0, \mathbf{y}_0)$ is the saddle point of $f(\mathbf{x}, \mathbf{y})$.

Cor. 2. *Let $\{v_{ij}\}$ be an $m \times n$ matrix. Then necessary and sufficient condition that $\{v_{ij}\}$ has a saddle point at $i = r, j = s$ is*

$$\min_{j} \max_{i} v_{ij} = a_{rs} = \max_{j} \min_{i} v_{ij} \qquad \text{...(19·56)}$$

Proof. Similar to *Corollary* 1. The matrix $\{v_{ij}\}$ remains as a real valued function of two variables i and j, then (19·56) immediately follows from (19·50).

Q. 1. Define a saddle point. State and prove the necessary and sufficient condition for a function $f(x, y)$ to possess a saddle point.

2. Let f be a function of two variables such then $f(x, y)$ is a real number whenever $x \in A$ and $y \in B$. Suppose that $\max\limits_{x \in A} \min\limits_{y \in B} f(x, y)$ and $\min\limits_{y \in B} \max\limits_{x \in A} f(x, y)$ both exist, then

prove that $\max\limits_{x \in A} \min\limits_{y \in B} f(x, y) \leq \min\limits_{y \in B} \max\limits_{x \in A} f(x, y)$,

What conclusion is drawn when

$\max\limits_{x \in A} \min\limits_{y \in B} f(x, y) \leq \min\limits_{y \in B} \max\limits_{x \in A} f(x, y)$, **[Delhi (OR.) 92]**

3. Discuss the saddle value problem and properties of optimal strategies in matrix games. **[Delhi (OR.) 95]**

SELF EXAMINATION QUESTIONS

1. What is a rectangular game ? Define 'pure strategy' and 'mixed strategy' in a game.
2. Explain the following terms : (i) pure strategy, (ii) Mixed strategy, and (iii) Optimal strategies.
3. Show how to solve a 2×2 two-person zero-sum game without any saddle point. Derive the expression for optimal strategies and the value of game.
4. Define mixed strategy and the value of a game in the theory of games. If a constant is added to each element of the payoff matrix, determine whether the set of optimal strategies of each player is the same or not. How does the value of the game then change ?
5. Explain what is meant by mixed extension of a rectangular game.
6. Define the expectation function in $m \times n$ rectangular game between two players. Enumerate and explain (without proof) the theorem for rectangular games in terms of the saddle point of the expectation function.
7. Let v be the value of a rectangular game with payoff matrix $B = (p_{ij})$.

(i) Show that $\min_i p_{ij} \le v \le \max_i p_{ij}$ and $\max_i \min_j p_{ij} \le v \le \min_j \max_i p_{ij}$.

(ii) For any unilateral deviation from optimal strategies, show that the expected yield will be unfavourable to the player who deviates from his optimal strategy.

(iii) If one of the players adheres to his optimal mixed strategy, show that the value of the game remains unaltered if the opponent uses the supporting strategies only, either or in a mixture.

8. Given the 2×2 payoff matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Suppose player A adopts the strategy (x, y), while B adopts the strategy (u, v) where x, y, u, v are all ≥ 0, s.t. $x + y = u + v = 1$.

(i) Express A's expected gain z in terms of x, y, u, v and a, b, c, d.

(ii) What is the effect on adding the same constant k to each element of the payoff matrix ?

(iii) What is the effect on z of multiplying each element of payoff matrix by the same constant k ?

(iv) How are the optimal strategies affected by these operations on payoff matrix.

9. (a) Show that the 2×2 game $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non-strictly determined if,

(i) $a < b$, $a < c$, $d < b$ and $d < c$ or (ii) $a > b$, $a > c$, $d > b$, $d > c$.

(b) If G is a $p \times q$ matrix game with optimal strategies p and q and the value v, what can you say about the optimal strategies and the value of the matrix game $mG + nE$? Prove your result.

Here $m > 0$, n is a real number and E is a $p \times q$ matrix with all its elements unity.

10. Prove that the following two theorems are equivalent.

(i) In an $m \times n$ rectangular game, a saddle point (x_0, y_0), $x_0 \in S_m$, $y_0 \in S_n$ always exists such that

$$E(x, y_0) \le E(x_0, y_0) \le E(x_0, y)$$

where S_r is the set of r-tuples of real non-negative numbers, subject to the condition that sum of the elements of each r-tuples is unity.

(ii) $E(\theta_i, y_0) \le E(x_0, y_0) \le E(x_0, \phi_j)$, where $1 \le i \le m$, $1 \le j \le n$, and $\theta_i \in S_m$, $\phi_j \in S_n$ such that one element of the m-tuple θ_i is unity and all others are zero, and similarly for ϕ_j.

11. If $X = \{x_i\}$ is optimal mixed strategy for A and $Y = \{y_j\}$ is optimal mixed strategy for B in a rectangular game specified by an $m \times n$ matrix $\{a_{ij}\}$, and V is the value of the game, then prove that if $E[e_i, Y] < V$, then $x_i = 0$, where e_i is the ith unit vector in n-dimensions, and $E[e_i, Y]$ is the expected amount received by A when he uses the strategy e_i and B uses the strategy Y.

12. Let $f(x, y)$ be real valued function defined for $x \in A$, $y \in B$, A and B being two sets.

Show that $\max_{x \in A} \min_{y \in B} f(x, y) \le \min_{y \in B} \max_{x \in A} f(x, y)$, provided that both exists. Further show that if (x_0, y_0) is a saddle point of $f(x, y)$, then

$$\max_{x \in A} \min_{y \in B} f(x, y) = (x_0, y_0) = \min_{y \in B} \max_{x \in A} f(x, y).$$

13. If all the elements of payoff matrix of a game are non-negative and every column of this matrix has at least one positive element, show that the value of the corresponding game is positive.

14. (a) Express a linear programming problem as a matrix game.

(b) What is a symmetric game ? Show that the value of a symmetric game is zero and that both players have identical optimal strategies. **[Delhi (OR.) 95, 93]**

15. "Game theory provides a systematic quantitative approach for analysing competitive situations in which the competitors make use of logical processes and techniques in order to determine an optimal strategy for winning" comment. **[Delhi MBA (Pt) 95]**

16. Give the comprehencive explanation of the term Game theory?

SELF EXAMINATION PROBLEMS

1. The payoff matrix for a 2-person, zero-sum game is given :

$$\begin{array}{cc} & \text{Player } A \\ & \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \text{Player } B \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \end{array} & \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \end{array}$$

Stating the criterion you adopt, find the optimal strategies for two players when a, b and c are all of the same sign. What simplifications are effected for the case, when a, b and c are not of the same sign ? What is the value of the game in either case ?

2. Given the following payoff table :

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & -1 & 2 \\ 4 & 4 & -3 \\ 0 & 3 & -4 \end{bmatrix} \end{array}$$

(i) Determine the value of th game, if possible.

(ii) What is the minimax criterion for stable games ?

(iii) What is minimax criterion for unstable game ?

(iv) Formulate the problem as a linear programming problem (No derivation of the problem is required. Just state the objective function, constraints etc.)

3. In a well-known children's game, each player says 'stone' or 'scissors' or 'papers'. If one says 'stone' and the other 'scissors' then the former wins a rupee. Similarly 'scissors' beats 'paper' and 'paper' beats 'stone', *i.e.*, the player calling the former word wins a rupee. If the two players name the same item, then there is a tie, *i.e.*, there is no payoff. Write down the payoff matrix and the L.P. problem of either of the two players. Find the value of the game and hence write down the optimal strategies of both players. **[Delhi (OR.) 90]**

[Hint. Let A be the row player and B be the column player. Then the payoff matrix for player A is :

$$\begin{array}{l} \\ \text{Stone} \\ \text{Paper} \\ \text{Scissors} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Stone} & \text{Paper} & \text{Scissors} \end{array} \\ \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \end{array}$$

4. The following matrix represents the payoff to P_1 in a rectangular game between two persons P_1 and P_2 :

$$\begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} 8 & 15 & -4 & -2 \\ 19 & 15 & 17 & 16 \\ 0 & 20 & 15 & 5 \end{bmatrix} \end{array}$$

By the notion of dominance, reduce the game to 2×4 game and then solve it graphically.
[Ans. (0, 15/16, 1/16), (0, 11/16, 0, 5/16), $v = 245/16$.]

5. Consider the following military situation. Side I is interested in bombing two areas, each containing a vast factory complex of side II. However, side I has only one bomber squadron to devote to this, and the entire squadron must be sent to only one target area in order to be effective. Side II can defend effectively against such an attack if it can send all of its available fighters to meet the bomber squadron. However, the two areas are so far apart that these fighters can be available for defending only one of the areas at any particular time.

If they are not defended by the fighters, the military value of bombing target area II is considered to be 2·5 times that of bombing target area I. However, if they are defended. the bombers must turn back with heavy losses without reaching the target. The military value of this loss is considered to be equal in magnitude to the gain if target area I were to be bombed undefended.

Use game theory to formulate this problem, and to determine the optimal mixed strategy of the respective side according to the minimax criterion.

6. Two items of value Rs. 100 and Rs. 130 are to be auctioned at a public sale. Only two bidders are interested in these items. Bidder A has Rs 100 available and bidder B has Rs. 80 available. What should be their strategies if each bidder is interested in maximizing his own gain.
[Hint. Proceed exactly as solved example]

7. Country A has two Ammuntion stores, one of which is twice as valuable as the other. B is an attacker who can destroy an undefended store but he can attack any one of them. A knows that B is about to attack one of the stores but does not know what should he do ? Note that A can successfully defend only one store at a time. What should A do to maximize his return ?
[Ans. A : (1/3, 2/3), $v = -2/3$]

8. The labour contract between your management and the union will terminate in the near future. A new contract must be negotiated preferably before the old one expires. You are a member of a management group charged with selecting a strategy for them during the coming negotiations. After a consideration of past experience, the group agrees that feasible strategies for the company and union are :

I. All out attack, hard aggressive bargaining.
II. A reasonably logical approach. III. A legalistic strategy.
IV. An agreeable conciliatory approach.

$$\begin{array}{cc} & \text{Union's Strategies} \\ & \begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{IV} \end{array} \\ \begin{array}{r} \text{Company's} \\ \text{Strategies} \end{array} \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \end{array} & \begin{bmatrix} 20 & 15 & 12 & 35 \\ 25 & 14 & 8 & 10 \\ 40 & 2 & 19 & 5 \\ 5 & 4 & 11 & 0 \end{bmatrix} \end{array}$$

The payoff table is given here. Find optimal strategy for the company. Determine the worth of your negotiations.
[Ans. Company (17/20, 0, 3/20, 0), Union (0, 7/20, 13/20, 0), $v = 261/20 = 13$ approx.]

9. Even through there are several manufactures of Scooters, two firms with brand name Janata and Praja, control their market in Western India. If both manufacturers make model changes of the same type for this market segment in the same year, their respective market shares remain constant. Likewise, if neither makes model changes, then also their market shares remain constant. The payoff matrix in terms of increased/decreased percentage market share under different possible conditions is given below :

$$\begin{array}{cc} & \begin{array}{cc} \text{Minor} & \text{Major} \end{array} \\ \begin{array}{r} \text{Minor} \\ \text{Major} \end{array} & \begin{bmatrix} 0 & 5 \\ 1 & 0 \end{bmatrix} \end{array}$$

		Praja		
		No change	Minor change	Major change
	No change	0	-4	-10
Janata	Minor change	3	0	5
	Major change	8	1	0

(i) Find the value of the game.
(ii) What change should Janata consider if this information is available only to itself ?
[Hint. This game has no saddle point. Making use of dominance principle, since the first row is dominated by the third one, we delete the first row. Similarly, first reduced payoff matrix then becomes :

10. A soft drink company calculated the market share of two products against its major competitor having three products and found out the impact of additional advertisement in any one of its product against the other. The payoff matrix is given below :

$$\begin{array}{cc} & \text{Competitor} \\ & \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \text{Company} \begin{array}{c} \text{I} \\ \text{II} \end{array} & \begin{bmatrix} 6 & 7 & 15 \\ 20 & 12 & 10 \end{bmatrix} \end{array}$$

What is the best strategy for the company as well as the competitor ? What is the payoff obtained by the company and the competitor in the long run ? Use graphical method or linear programming method to obtain the solution.

11. Two players are each provided with an ace of diamonds and an ace of clubs. Player P_1 is also given the two of diamonds and player P_2 , the two of clubs. In the first move, P_1 shows of his cards and P_2 ignorent of P_1's choice, shows one of his cards. P_1 wins if the suits match and P_2 wins if they do not. The amount of payoff is the numerical value of the card shown by the winner. If both the twos are shown, the payoff is zero.
[**Ans.** (1/2, 1/2), (1/2, 1/2), $v = 0$.]

12. An enterprising young statistician believes that he has developed a system for winning a popular Lasvegas game. His colleagues do not believe that this is possible, so they have made a large bet with him. They bet that, starting with three chips, he will not have five chips after three plays of the game. Each play of the game involves betting any desired number of available chips and then either winning or losing this number of chips. The statlistitian believes that his system will give him a probability of 2/3 of winning a given play of the game. Assuming he is correct, determine his optimal policy regarding how many chips to bet (if any) at each of the three plays of the game. The decision at each play should take into account the results of earlier plays. The objective is to maximize the probability of winning his bet with his colleagues.

13. Consider a modified form of "matching biased coins" game problem. The matching player is paid Rs. 8.00 if the two coins turn both heads and player is paid Rs. 3 when the two coins do not match. Given the choice of being the matching or non-matching player, which one would you choose and what would be your strategy ?
[**Ans.** (4/15, 11/15), (4/15, 11/15), $v = -1/15$]

14. Two firms A and B are competing for an increased market share. To improve its market share, both the firms decide to employ the following promotional strategies :

(A_1, B_1) = No promotion, (A_2, B_2) = Moderate promotion, (A_3, B_3) = Much promotion. The payoff matrix, shown in the following table, describe the increase in market share for firm A and decrease in market share for firm B :

		Firm B		
		B_1	B_2	B_3
Firm A	A_1	5	20	−10
	A_2	10	6	2

Determine the optimal strategies for each firm and the value of the game. **[Delhi MBA. (RT). 95]**

15. Two computer manufacturers A and B are attempting to sell computer systems to two banks 1 and 2. Company A has 4 salesmen, company B has only 3 available. The computer companies must decide upon how many salesmen to assign to sell on each bank. Thus company A can assign 4 salesmen to assign to bank 1 and none to bank 2 or three to bank 1 and one to bank 2, etc.

Each bank will buy one computer system. The probability that a bank will buy from a particualr computer company is directly related to the number of salesmen calling from that company relative to total salesmen calling. Thus, if company A assignes three men to bank 1 and company B assigns two men the odds would be three out of five that bank 1 would purchase company A's computer system. (if none calls from either company the odds are one-half for buying either computer.

Let the pay-off be the expected number of computer systems that company A sells. (Then 2 minus this pay-off is the expected number company B sells).

What strategy would company A use in allocating its salesmen ? What strategy should company B use ? What is the value of the game to company A ? What is the meaning of the value of the game in this problem ? **[Delhi (MBA) April 95]**

[**Ans.** The payoff matrix for company A :

$$\begin{array}{ccc} & & \text{Company } B \\ & & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \text{Company } A & \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} & \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 1 & 1/2 & 1/3 & 1/4 \\ 1 & 2/3 & 2/4 & 2/5 \\ 1 & 3/5 & 3/5 & 3/6 \\ 1 & 4/5 & 4/6 & 4/7 \end{bmatrix} \end{array}$$

Optimum strategy for A is A_5 and for B is B_4.

Value of the game = 4/7, *i.e.*, prob. of success for A is 4/7 or 51% supply.

16. Assume that two firms are competing for market share for a particular product. Each firm is considering what promotional strategy to employ for the coming period. Assume that the following pay-off matrix describes the increase in market share of Firm A and the decrease in market share for Firm B. Determine the optimum strategies for each firm.

		Firm B No promotion	Moderate promotion	Much promotion
	No promotion	5	0	− 10
Firm A	Moderate promotion	10	6	2
	Much promotion	20	15	10

(i) Which firm would be winner, in terms of market share ?
(ii) Would the solution strategies necessarily maximize profits for either of the firms ?
(iii) What might the two firms do to maximize their profits ? **[Delhi (M.B.A.) Nov. 97]**
[**Ans.** Optimum strategy for both *A* and *B* is much promotive ? Value of game = 10].

17. (a) Solve the following 2-person, zero-sum game :

Player B

$$\textbf{Player } A \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} 10 & 5 & 7 \\ 6 & 7 & 5 \\ 7 & 6 & 7 \end{bmatrix} \end{array}$$

[Delhi (M. Com.) 98]

(b) The pay-off matrix for a two person zero-sum game is given below. Find the best strategy for each player and the value of the game.

Player B

$$\textbf{Player } A \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{ccccc} B_1 & B_2 & B_3 & B_4 & B_5 \end{array} \\ \begin{bmatrix} -2 & 0 & 0 & 5 & 3 \\ 3 & 3 & 1 & 2 & 2 \\ -4 & -3 & 0 & -2 & 6 \\ 5 & 3 & -4 & 2 & -6 \end{bmatrix} \end{array}$$

18. Two leading firms (firm *A* and firm *B*), for year's have been selling suitings, which is but a small part of both firm's total sales. The Marketing Director of firm *A* raised the question. "What should his firm's strategies be in terms of advertising for the product in question ?" The systems group of the firm *A* developed the following data for varying degrees of advertising :

(i) No advertising, medium advertising and large advertising for both firms will result in equal market share.
(ii) Firm *A* with no advertising : 40 per cent of the market with medium advertising by firm *B* and 28 per cent of the market with large advertising by firm *B*.
(iii) Firm *A* using medium advertising : 70 per cent of the market with no advertising by the firm *B* and 45 per cent of the market with large advertising by firm *B*.
(iv) Firm *A* using large advertising : 75 per cent of the market with no advertising by firm *B*.

Based upon the above information, answer the marketing director's question. **[Sadar Patel (M.B.A.) 97]**

19. Solve the following game :

Player B

$$\textbf{Player } A \quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} I & II & III & IV & V & VI \end{array} \\ \begin{bmatrix} 4 & 2 & 0 & 2 & 1 & 1 \\ 4 & 3 & 1 & 3 & 2 & 2 \\ 4 & 3 & 7 & -5 & 1 & 2 \\ 4 & 3 & 4 & -1 & 2 & 2 \\ 4 & 3 & 3 & -2 & 2 & 2 \end{bmatrix} \end{array}$$

[Gujarat (M.B.A.) 97]

[**Hint.** Game has no saddle point. Use principle of dominance.
Ans. S_A = [0, 6/7, 1/7, 0, 0], S_B = [0, 0, 4/7, 3/7, 0, 0], v = 13/7]

20. Two leading firms *A* and *B* are planning to make fund allocation for advertising their product. The matrix given below shows the percentage of market shares of firm *A* and *B* for their various advertising policies :

		Firm B No advertising	Medium advertising	Heavy advertising
	No advertising	60	50	40
Firm A	Medium advertising	70	70	50
	Heavy advertising	80	60	75

Find the optimum strategies for the two firms and the expected outcome when both the firms follow their optimum strategies. **[H.P. (M.B.A.) Jan. 99]**

[**Hint.** No saddle point. Use dominance. Solve the 2 × 2 game $\begin{pmatrix} 70 & 50 \\ 60 & 75 \end{pmatrix}$]

21. Obtain the optimum strategies for both the firms for the adjoining pay-off matrix :

$$\begin{array}{cc} & \textbf{Firm } B \\ \textbf{Firm } A & \begin{bmatrix} 12 & 10 & 8 \\ 14 & 14 & 10 \\ 16 & 12 & 15 \end{bmatrix} \end{array}$$

[A.I.M.A. (P.G. Dip. in Management), Dec. 96]

[**Hint.** No saddle point. Use dominance.
Ans. S_A = [0, 3/7, 4/7], S_B = [0, 5/7, 2/7], v = 90/7]

22. Assume that two firms are competing for market share for a particular product. Each firm is considering what promotional strategy to employ for the coming period. Assume that the following pay-off matrix describes the increase in market share for for Firm A and the decrease in market share for Firm B. Determine the optimum strategies for each firm.

		Firm B No promotion	Firm B Moderate promotion	Firm B Much promotion
Firm A	No promotion	5	0	− 10
	Moderate promotion	10	6	2
	Much promotion	20	15	10

(i) Which firm would be the winner, in terms of market share ?
(ii) Would the solution strategies necessarily maximize profits for either of the firms ? **[Delhi (M.B.A.) April 99]**

23. Two candidates, X and Y, are competing for the councillor's seat in a city municipal corporation, and X is attempting to increase his total votes at the expense of Y. The strategies available to each candidate involve personal contacts, newspaper insertions/speeches or television appearance/advertising. The increase in votes available to X given various combinations of strategies are given below. (Assume that this is a zero-sum game, *i.e.*, any gain of X is equal to the votes lost by Y). Determine the optimum strategies that should be adopted by X during his election compaign. How many votes should X gain by the following optimum strategy ?

		Y Personal contacts	Y Newspapers	Y Television
X	Personal contacts	30,000	20,000	10,000
	Newspapers	60,000	50,000	25,000
	Television	20,000	40,000	30,000

[Delhi (M.B.A.) Dec. 95]

24. Two separate firms (A and B) have for years been selling a competing product which forms a part of both firm's total sales. The marketing executive of firm A raised the question. "What should be the firm's strategies in terms of advertising for the product in question." The market research team of firm A developed the following data for varying degrees of advertising :

(i) No advertising, medium advertising, and large advertising for both firms will result in equal market shares.

(ii) Firm A with no advertising : 40% of the market with medium advertising by firm B and 28% of the market with large advertising by firm B.

(iii) Firm A using medium advertising : 70% of market with no advertising by firm B and 45% of the market with large advertising by firm B.

(iv) Firm A using large advertising : 75% of the market with no advertising by firm B and 47·5% of the market with medium advertising by firm B.

(a) Based upon the foregoing information, answer the marketing executive's question.
[Delhi (M.B.A.) Dec. 96; March 99]

(b) What advertising policy should firm A pursue when consideration is given to the above factors : selling price, Rs. 4·00 per unit; variable cost of product, Rs. 2·50 per unit; annual volume of 30,000 units for firm A ; cost of annual medium advertising Rs. 5,000 and cost of annual large advertising Rs. 15,000 ? What contribution, before other fixed costs, is avaialble to the firm ? **[A.I.M.A. (P.G. dip. in Management), 97]**

25. A soft drink company calculated the market share of two products against its major competitor having three products and found out the impact of additional advertisment in any one of its products against the other.

$$\begin{array}{cc} & \textbf{Competitor } B \\ & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \textbf{Company } A \; \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 6 & 7 & 15 \\ 20 & 12 & 10 \end{bmatrix} \end{array}$$

What is the best strategy for the company as well as the competitor ? What is the pay-off obtained by the company and the competitor in the long run ? Use graphical method to obtain the solution. **[Delhi (M.B.A.) April 98]**
[**Ans.** Company A = [2/3, 1/3, 0], Competitor B = [7/12, 5/12], v = 1/3]

26. Firm X is fighting for its life against the determination of firm Y to drive it out of the industry. Firm X has the choice of increasing price, leaving it unchanged, or lowering it. Firm Y has the same three options. Firm X's gross sales in the event of each of the pairs of choices are shown below :

		Firm Y's pricing strategies		
		Increase price	Do not change	Reduce price
Firm X's	Increase price	90	80	110
pricing	Do not change	110	100	90
strategies	Reduce price	120	70	80

Assuming firm X as the miximizing one, formulae the problem as a linear programming problem.

[Osmania (M.B.A.) Nov. 96]

[**Ans.** Player A : (3/8, 13/24, 1/12), Player B : (7/24, 5/9, 11/22), v = 91/24].

27. In zero-sum two person children's game of stone, paper and scissors, both players simultaneously call out stone, paper or scissors. The combination of paper and stone is a win of one unit for player calling paper (paper covers stone); stone and scissors is a win for stone (stone breaks scissors), and scissors and paper is a win for scissors (scissors cut paper). A call of the same item represents no pay-off. Write the pay-off matrix and the equivalent linear programign problem to the above game. Find the optimum strategy for both the players and the value of the game.

[A.I.M.A. (P.G. Dip. in Management), Dec. 95]

28. In a town, there are only two discount stores ABC and XYZ. Both stores run annual pre-diwali sales during the first week of October. Sales are advertised through local newspapers with the aid of an advertising firm. ABC stores constructed following pay-off in units of Rs. 1,00,000. Find the optimum strategies for both stores and the value of the game ;

Strategies of XYZ

$$\textbf{Strategies of ABC}\quad \begin{array}{c|c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & -2 & 1 \\ -1 & 3 & -2 \\ -1 & -2 & 3 \end{bmatrix} \end{array}$$

[Bombay (M.M.S.) 95]

29. Determine the saddle-point solution, the associated pure strategies and the value of the game whose pay-off matrix is given below :

	B_1	B_2	B_3	B_4
A_1	4	−4	−5	6
A_2	−3	−4	−9	−2
A_3	6	7	−8	−9
A_4	6	3	−9	5

[Meerut (MCA II) 2000]

30. Solve the game approximately :

$$A\quad \begin{array}{c|c} & \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \end{array} & \begin{bmatrix} -1 & 2 & 1 \\ 1 & -2 & 2 \\ 3 & 4 & -3 \end{bmatrix} \end{array}$$

[Meerut 2002]

31. Use dominance principle to simplify the rectangular game with the following payoff matrix and then solve graphically.

		Player B			
		I	II	III	IV
Player A	1	18	4	6	4
	2	6	2	13	7
	3	11	5	17	3
	4	7	6	12	2

[AIMS (MBA) 2002]

32. (a) Use the Dominance principle and solve the game :

		B		
		1	2	3
A	I	1	−3	−2
	II	0	−4	2
	III	−5	2	3

[VTU 2003]

(b) Solve the following game using graphical method.

		B		
		1	2	3
A	I	3	−1	0
	II			
	III	2	1	−1

[VTU 2003]

OBJECTIVE QUESTIONS

1. Two-person zero-sum game means that the
   (a) sum of losses to one player equals the sum of gains to other.
   (b) sum of losses to one player is not equal to the sum of gains to other.
   (c) both (a) and (b).
   (d) none of the above.
2. Game theory models are classified by the
   (a) number of players. (b) sum of all payoffs. (c) number of strategies. (d) all of the above.
3. A game is said to be fair, if
   (a) both upper and lower values of the game are same and zero.
   (b) upper and lower values of the game are not equal.
   (c) upper value is more than lower value of the game.
   (d) none of the above.
4. What happens when maximin and minimax values of the game are same?
   (a) No solution exists. (b) Solution is mixed. (c) Saddle point exists. (d) None of the above.
5. A mixed strategy game can be solved by
   (a) algebraic method. (b) matrix method. (c) graphical method. (d) all of the above.
6. The size of the payoff matrix of a game can be reduced by using the principle of
   (a) game inversion. (b) rotation reduction.
   (c) dominance. (d) game transpose.
7. The payoff value for which each player in a game always selects the same strategy is called the
   (a) saddle point. (b) equilibrium point.
   (c) both (a) and (b). (d) none of the above.
8. Games which involve more than two players are called
   (a) conflicting games. (b) negotiable games.
   (c) n-person games. (d) all of the above.
9. When the sum of gains of one player is equal to the sum of losses to another player in a game, this situation is known as
   (a) biased game. (b) zero-sum game. (c) fair game. (d) all of the above.
10. When no saddle point is found in a payoff matrix of a game, the value of the game is then found by
    (a) knowing joint probabilities of each row and column combination to calculate expected payoff for that combination and adding all such values.
    (b) reducing size of the game to apply algebraic method.
    (c) both (a) and (b).
    (d) none of the above.

Answers

1. (c) **2.** (d) **3.** (a) **4.** (c) **5.** (d) **6.** (c) **7.** (c) **8.** (c) **9.** (b) **10.** (c).

***

20

INVENTORY/PRODUCTION MANAGEMENT-I
(Deterministic Inventory Models)

20.1. INTRODUCTION

In our daily life, we observe that a small retailer knows roughly the demand of his customers in a month or a week, and accordingly places orders on the wholesaler to meet the demand of his customers. But, this is not the case with a manager of a big departmental store or a big retailer, because the stocking in such cases depends upon various factors, *e.g.* demand, time of ordering, lag between orders and actual receipts, etc. So the real problem is to have a compromise between ***over-stocking and under-stocking.***

The study of such type of problems is known by the term ***'Material Management'*** or ***'Inventroy Control'.*** The inventory control may be defined as follows :

Definition. *The function of directing the movement of goods through the entire manufacturing cycle from the requisitioning of raw materials to the inventory of finishes goods orderly mannered to meet the objectives of maximum customer-service with minimum investment and efficient (low-cost) plant operation.*

The models discussed in this chapter will be limited mainly to the elementary type, because the analytical study of the other cases becomes more difficult. After a general discussion of each indicated type of model, we shall give many interesting solved examples so that all the necessary ideas may be clear to the students. We shall also discuss another class of inventory models, namely ***'Inventory Models with Price Breaks'*** (*i.e.* ***Quantity Discount Models***).

20.2. WHAT IS INVENTORY ?

In broad sense, ***inventory*** may be defined as the stock of goods, commodities or other economic resources that are stored or reserved in order to ensure *smooth* and *efficient* running of business affairs.

The inventory or stock of goods may be kept in any of the following forms :

(i) ***Raw material inventory,*** *i.e.* raw materials which are kept in stock for using in the production of goods.

(ii) ***Work-in-process inventory,*** *i.e.* semifinished goods or goods in process which are stored during the production process.

(iii) ***Finished goods inventory,*** *i.e.* finished goods awaiting shipment from the factory.

(iv) Inventory also include : *furniture, machinery, fixtures*, etc.

The term inventory may be classified in two main categories.

1- Direct Inventories

The items which play a direct role in the manufacture and become an integral part of finished goods are included in the category of *direct inventories.* These may be further classified into four main groups :

(a) ***Raw material inventories*** are provided :
   (*i*) for economical bulk purchasing, (*ii*) to enable production rate changes
   (*iii*) to provide production buffer against delays in transportation, (*iv*) for seasonal fluctuations.

(b) ***Work-in-process inventories*** are provided :
   (*i*) to enable economical lot production, (*ii*) to cater to the variety of products
   (*iii*) for replacement of wastages, (*iv*) to maintain uniform production even if amount of sales may vary.

(c) ***Finished-goods inventories*** are provided :
(i) for maintaining off-self delivery, *(ii)* to allow stabilization of the production level *(iii)* for sales promotion.

(d) ***Spare parts.***

2 - Indirect Inventories

Indirect inventories include those items which are necessarily required for manufacturing but do not become the component of finished production, like : *oil, grease, lubricants, petrol, office-material, maintenance material,* etc.

20.3. TYPES OF INVENTORY MODELS

Basically, there are five types of inventory models :

(i) Fluctuation Inventories. These have to be carried because sales and production times cannot be predicted accurately. In real-life problems, there are fluctuations in the demand and lead-times that affect the production of items. Such type of reserve stocks or safety stocks are called *fluctuation inventories.*

(ii) Anticipation Inventories. These are built up in advance for the season of large sales, a promotion programme or a plant shut-down period. In fact, anticipation inventories store the men and machine hours for future requirements.

(iii) Cycle (lot-size) Inventories. In practical situations, it seldom happens that the rate of consumption is the same as the rate of production or purchasing. So the items are procured in larger quantities than they are required. This results in cycle (or lot-size) inventories.

(iv) Transportation Inventories. Such inventories exist because the materials are required to move from one place to another. When the transportation time is long, the items under transport cannot be served to customers. These inventories exist solely because of transportation time.

(v) Decoupling Inventories. Such inventories are needed for meeting out the demands during the decoupling period of manufacturing or purchasing.

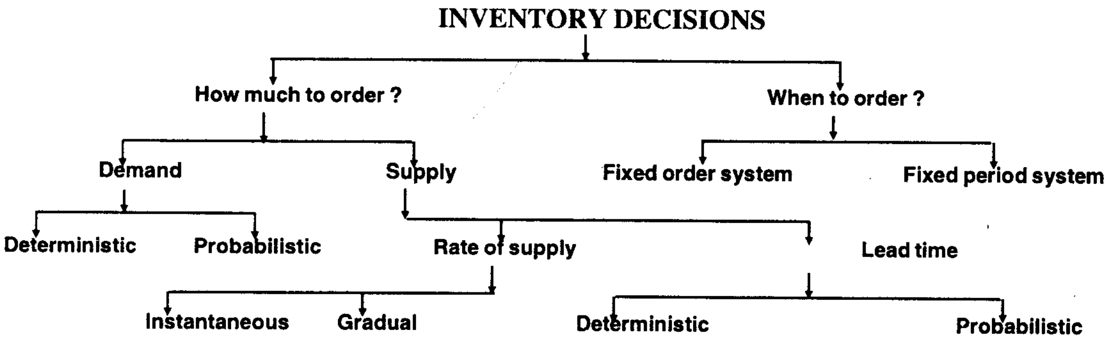
20.4. INVENTORY DECISIONS

The managers must take two basic decisions in order to accomplish the functions of inventory. The decisions made for every item in the inventory are :

(i) *How much amount of an item should be ordered when the inventory of that item is to be replenished ?*

(ii) *When to replenish the inventory of that item ?*

Inventory decisions may be classified as follows :



Before taking inventory decisions, it is necessary to develop an inventory model.

Q. 1. Define Inventory. What are the different types of inventory in industries ? Why is it important to control inventory ?
[JNTU (B. Tech.) 2003; Meerut 2002; Agra 99; Kanpur M.Sc (Math.) 96]

2. What is Inventory Management ? Briefly explain the major decisions concerning inventory.

3. What is inventory problem ? **[JNTU (Mech & Prod.) 2004; Garhwal M.Sc. (Stat) 93]**

20.5. HOW TO DEVELOP AN INVENTORY MODEL ?

As explained earlier, inventory models are concerned with two main decisions : *how much to order at a time* and *when to order so as to minimize the total cost* ? The sequence of basic steps required for developing an inventory model may be organized as follows :

***Step* 1.** First take the physical stock of all the inventory items in an organization.

***Step* 2.** Then, classify the stock of items into various categories. Although several methods are available to classify the inventories; but the selected method must serve the objectives of inventory management. For example, inventory items may be classified as *raw materials, work-in process, purchased components, consumable stores and maintenance spares*, and *finished goods*, etc.

***Step* 3.** Each of above classifications may be further divided into several groups. For example, consumable stores and maintenance spares can be further divided into the following groups :
(i) *building materials*, (ii) *hardware items*, (iii) *lubricants and oils*, (iv) *textiles and fibres*, (v) *electric spares*, (vi) *mechanical spares*, (vii) *stationary items*, etc.

***Step* 4.** After classification of inventories, each item should be assigned a suitable code. Coding system should be flexible so that new items may also be permitted for inclusion.

***Step* 5.** Since the number of items in an organization is very large, separate inventory management model should be developed for each category of items.

***Step* 6.** Use A-B-C or V-E-D classification (as discussed in the next chapter) which provide a basis for a selective control of inventories through formulation of suitable inventory policies for each category.

***Step* 7.** Now decide about the inventory model to be developed. For example, fixed-order-quantity system may be developed for 'A' class and high valued 'B' class items, whereas periodic review system may be developed for low valued 'B' class and 'C' class items.

***Step* 8.** For this, collect data relevant to determine ordering cost, shortage cost, inventory carrying cost, etc.

***Step* 9.** Then, make an estimate of annual demand for each inventory item and their prevailing market price.

***Step* 10.** Estimate *lead-time, safety stock* and *reorder level*, if supply is not instantaneous. Also, decide about the service-level to be provided to the customers.

***Step* 11.** Now develop the inventory mode.

***Step* 12.** Finally, review the position and make suitable alterations, if required, due to current situations or constraints.

Before we proceed to discuss inventory models, it is very desirable to consider briefly the costs involved in the inventory decisions.

20.6. COSTS INVOLVED IN INVENTORY PROBLEMS

1. Holding Cost (C_1 or C_h). The cost associated with carrying or holding the goods in stock is known as ***holding*** or ***carrying*** cost which is usually denoted by C_1 or C_h per unit of goods for a unit of time. Holding cost is assumed to vary directly with the size of inventory as well as the time for which the item is held in stock. The following components constitute the holding cost :

(i) ***Invested Capital Cost.*** This is the interest charge over the capital investment. Since this is the most important component, a careful investigation is required to determine its rate.

(ii) ***Record-Keeping and Administrative Cost.*** This signifies the need of keeping funds for maintaining the records and necessary administration.

(iii) ***Handling Costs.*** These include all costs associated with movement of stock such as : *cost of labour, over-head cranes, gantries* and *other machinery* required for this purpose.

(iv) ***Storage Costs.*** These involve the rent of storage space or depreciation and interest even if the own space is used.

(v) ***Depreciation, Deterioration and Obsolescence Costs.*** Such costs arise due to the items in stock being out of fashion or the items undergoing chemical changes during storage (*e.g.* rusting in steel).

(vi) ***Taxes and Insurance Costs.*** All these costs require careful study and generally amounts to 1% to 2% of the invested capital.

(vii) ***Purchase Price or Production Costs.*** Purchase price per-unit item is affected by the quantity purchased due to *quantity discounts* or *price-breaks*. Production cost per unit item depends upon the length of production runs. For long smooth production runs this cost in lower due to more efficiency of men and machines. So the order quantity must be suitably modified to take the advantage of these price discounts.

If P is the purchase price of an item and I is the stock holding cost per unit item expressed as a fraction of stock value (in rupees), then the holding cost $C_1 = IP$.

(viii) ***Salvage Costs or Selling Price.*** When the demand for an item is affected by its quantity in stock, the decision model of the problem depends upon the profit maximization criterion and includes the revenue (sales tax etc.) from the sale of the item. Generally, salvage costs are combined with the storage costs and not considered independently.

2. Shortage Costs or Stock-out Costs (C_2 or C_s). The penalty costs that are incurred as a result of running out of stock (*i.e.*, shortage) are known as ***shortage*** or ***stock-out*** **costs**. These are denoted by C_2 or C_s per unit of goods for a specified period.

These costs arise due to **shortage of goods, sales may be lost, good-will may be lost** either by a delay in meeting the demand or being quite unable to meet the demand at all. In the case where the unfilled demand for the goods can be satisfied at a latter date (backlog case), these costs are usually assumed to vary directly with the shortage quantity and the delaying time both. On the other hand, if the unfilled demand is lost (no backlog case), shortage costs become proportional to shortage quantity only.

3. Set-up Costs (C_3 or C_o). These include the fixed cost associated with obtaining goods through *placing of an order* or *purchasing* or *manufacturing* or *setting up a machinery* before starting production. So they include costs of purchase, requisition, follow-up, receiving the goods, quality control, etc. These are also called *order costs* or replenishment costs, usually denoted by C_3 or C_o per production run (cycle). They are assumed to be independent of the quantity ordered or produced.

Q. 1. Explain clearly the various costs that are involved in inventory problems with suitable examples. How they are inter-related ? **[JNTU (MCA III) 2004; Meerut (Stat.) 98, 95, 90; (Math) 96, (Stat.) 95; Garhwal M.Sc. (Stat.) 92; Nagpur (MBA) 90]**

2. Explain in detail, what constitutes the ordering cost and carrying cost ? With the help of a graph show how they behave with the increase in order quantity. **[JNTU (Mech. & Prod.) 2004]**

3. What is an Inventory System ? Explain clearly the different costs that are involved in inventory problems with suitable examples. **[Meerut (Stat.) 95, (Math.) 90; Kanpur M.Sc. (Math.) 93]**

4. What are the categories of costs that are associated with in developing a sound inventory model ? What are the components of cost under each of them ? **[JNTU (B. Tech.) 2003; Agra 93]**

5. What are the different inventory costs associated with inventory control ? **[Bhubnashwar (IT) 2004]**

20.6-1. Why Inventory is Maintained ?

As we are aware of the fact that the inventory is maintained for efficient and smooth running of business affairs. If a manufacturer has no stock of goods at all, on receiving a sale-order he has to place an order for purchase of raw materials, wait for their receipt and then start his production. Thus, the customers will have to wait for a long time for the delivery of the goods and may turn to other suppliers. This results in a heavy loss of business. So it becomes necessary to maintain an inventory because of the following reasons :

(i) *Inventory helps in smooth and efficient running of business.*

(ii) *Inventory provides service to the customers immediately or at a short notice.*

(iii) *Due to absence of stock, the company may have to pay high prices because of piece-wise purchasing. Maintaining of inventory may earn price discount because of bulk-purchasing.*

(iv) *Inventory also acts as a buffer stock when raw materials are received late and so many sale-orders are likely to be rejected.*

(v) *Inventory also reduces product costs because there is an additional advantage of batching and long smooth running production runs.*

(vi) *Inventory helps in maintaining the economy by absorbing some of the fluctuations when the demand for an item fluctuates or is seasonal.*

(vii) *Pipeline stock (also called process and movement inventories) are also necessary where the significant amount of time is consumed in the trans-shipment of items from one location to another.*

Mathematically, the problem of maintaining the inventory arises due to the fact that—if a person (*e.g.*, a big retailer) decides to have a large stock, his holding cost C_1 increases but his shortage cost C_2 and set-up cost C_3 decrease. On the other hand, if he has small stock, his holding cost C_1 decreases but shortage cost C_2 and set-up cost C_3 increase. Similarly, if he decides to order very frequently, his *ordering* cost increases while other costs may decrease. So it becomes necessary to have a compromise between over-stocking and under-stocking by making optimum (most favourable) decisions by controlling the value of some variables which are at our disposal.

Q. 1. What are the advantages and disadvantages of increased inventory ? Briefly explain the objectives that must be fulfilled by an Inventory Contract System. **[Meerut 2002]**

2. Why inventory is maintained ? **[Meerut 95; Kanpur 93]**

20.7. VARIABLES IN INVENTORY PROBLEM

We shall now proceed to classify the variables which are involved in an inventory problem.

The variables used in any inventory model are of two types :

(a) *Controlled variables*, (b) *Uncontrolled variables.*

Controlled Variables

The following are the variables that may be controlled separately or in combination :

1. *How much quantity acquired (by purchase, production, or some other means).*

This may be adjusted for each type of resources separately or for all items collectively in one of the following ways :

(i) *The quantity to be ordered should be q quantity units ;*

(ii) *The quantity to be ordered should be such as to raise the stock level to S quantity units;*

(iii) *The quantity to be ordered should be such as to raise the stock level on hand and on order to z.*

2. *The frequency or timing of acquisition. How often or when to replenish the inventory ?*

The inventory should be replenished when—

(i) *the amount in stock is equal to or below S quantity units;*

or (ii) *the amount in stock and the amount of order are equal to or below z;*

or (iii) *at every t time units.*

3. *The completion stage of stocked items.*

More finished the goods, lesser the delay in meeting the demands. But, on the other hand higher will be the cost of holding them in stock. Lesser finished the stock items, longer the time in meeting the demands, consequently lesser the cost of holding in stock.

Most of the inventory models involve only first two types of controlled variables.

Uncontrolled Variables

The following are the principal variables that may be controlled :

1. *The holding costs (C_1), shortage or penalty costs (C_2), set-up costs (C_3).*

2. *Demand (the number of items required per period).*

We note that it is not necessarily the amount sold, because some demand may go unfilled because of shortages or delays. It is, in fact, the demand that would be sold if all that is required were available. The demand pattern of items may be either *deterministic* or *probabilitic.*

In the *deterministic* case, it is assumed that quantities needed over subsequent periods of time are known exactly. Further, the known demand may be *fixed* or *variable* with time. Such demands are called *static* and *dynamic* respectively.

The *probabilistic* demand occurs when the demand over a certain period of time is not known with certainty; but its pattern can be described by a known probability distribution. A probabilistic demand may be either *stationary* or *non-stationary* over time.

3. ***Lead time (the time between placing an order and its arrival in stock).*** If the *lead time* is known and is not equal to zero, and if demand is deterministic, all that one requires to do is to *order in advance* by an amount of time *equal* to the *lead time*. While there is no need to order in advance, if the lead time is *zero*.

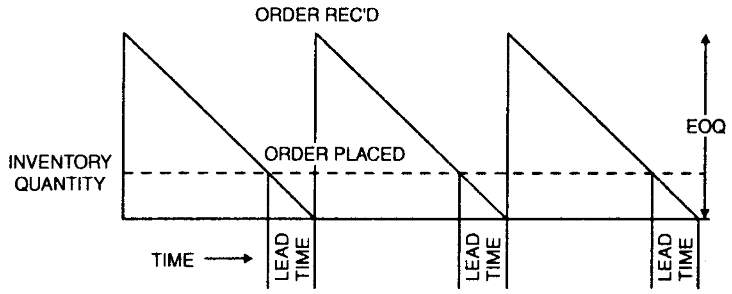


Fig. 20.1. Inventory with constant demand rate and constant lead time.

If, however, the lead time is a variable that is known only probabilistically, the question of when to order is more difficult one. If either the demand or the lead time is known only probabilistically, the amount and the timing of replenishment are found by considering *expected costs* of holding and shortage over the lead time required.

4. ***Amount delivered (supply of goods).*** The supply of goods may be instantaneous or spread over a period of time. If a quantity q is ordered for purchase or production, the amount delivered may vary around q with a known probability density function.

20.8. CLASSIFICATION OF CHARACTERISTICS OF INVENTORY SYSTEMS

We may classify the characteristics of inventory probelms as follows :

(1) Holding cost C_1 per unit item — Constant

(2) Shortage cost C_2 per unit time

(3) Set-up cost C_3 per cycle — Variable

(4) Demand
- Known
  - Constant (static)
  - Variable (dynamic)
- Estimated

(5) Quantities required
- Discrete units
- Continuous units

(6) Distribution of withdrawals over time
- Continous — Constant rate
- Discontinuous — Variable rate

(7) Reorder lead time
- Virtually zero
- Positive

(8) Reorder cycle time
- Known — Constant
- Estimated — Variable

(9) Input quantities
- Discrete — Constant
- Continuous — Variable

(10) Distribution of inputs over time
- Continuous — Constant rate
- Discontinuous — Discontinuous rate

Q. 1. Describe the basic characteristics of an inventory system. **[IGNOU 2000]**

2. Explain with suitable examples, fixed order quantity and fixed interval system of inventory management. **[JNTU (Mech. & Prod.) 2004]**

20.9. A LIST OF SYMBOLS USED

The following symbols are generally used in connection with the inventory models presented in this chapter :

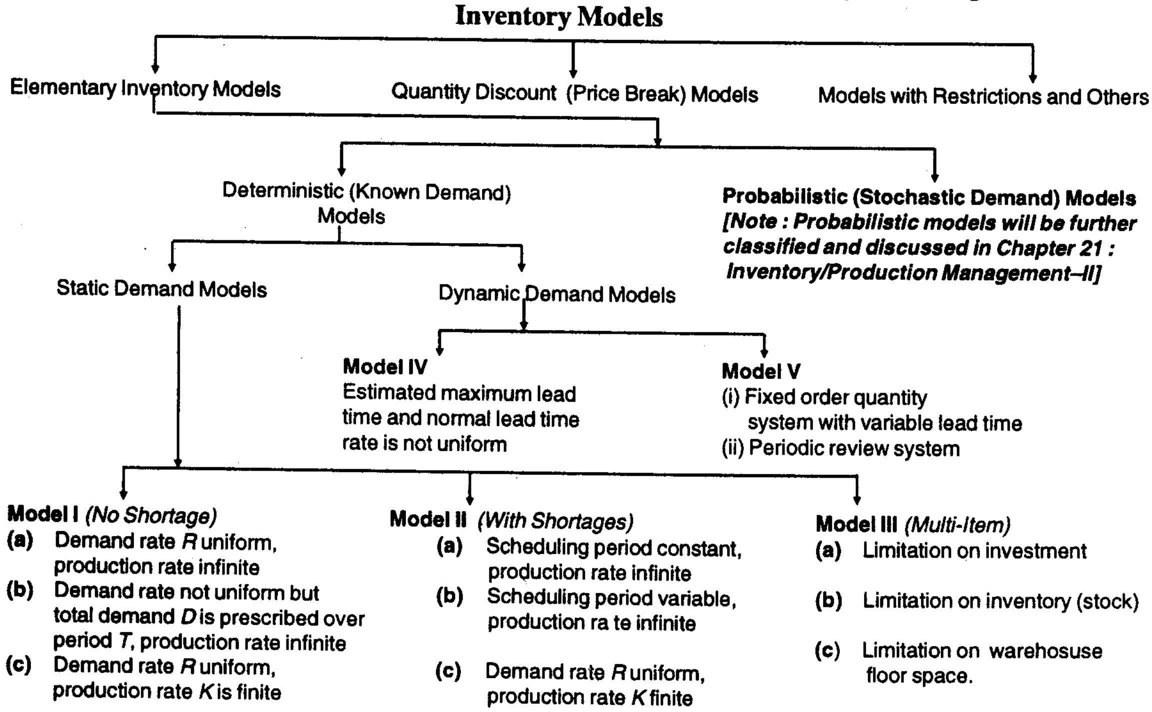
C_1 = holding cost per quantity unit per unit time
C_2 = shortage cost per quantity unit per unit time for the back-log case and per unit item only for no back-log case
C_3 = set-up (ordering) cost per order
R = demand rate
K = production rate
t = scheduling time period which is variable
t_p = prescribed scheduling time period
x = demand during prescribed period t_p with probability $F(x)$
y = demand during lead time L with probability $G(x)$
z = order level (or stock level)
D = total demand or annual demand
q = quantity already present in the beginning
L = lead time
$f(x)$ = probability density function for demand x.

Note. Sometimes the costs C_1, C_2 and C_3 are also denoted by C_h, C_s and C_o respectively.

20.10. CLASSIFICATION OF INVENTORY MODELS

Now, we may classify the inventory models with respect to above characteristics (in **Sec. 20.8**) of inventory problems. This classification can be represented in a systematic order by a chart as given below :

Inventory Models



Q. 1. What are inventory models ? Enumerate various types of inventory models and describe them briefly.
[JNTU (B. Tech.) 2003; Kanpur M.Sc.(Math.) 96; Garhwal M.Sc. (Stat) 91]

2. Give a brief summary of the various models for stock control problem and discuss their uses.

3. What is meant by inventory problem ? Give the classification of different inventory models. **[Garhwal M.Sc (Stat.) 95]**

4. Explain inventory control system. Describe the factors involved in Inventory Analysis. **[Bhubnashwar (IT) 2004]**

I – Deterministic Elementary Inventory Models

20.11. CONCEPT OF AVERAGE INVENTORY

For developing the economic lot size inventory model, following assumptions must be made regarding the purchase of a single item of inventory.

(i) Firstly, demand for the item is at a constant rate and is known to the decision maker in advance.

(ii) Secondly, the lead time (which is the elapsed time between the placement of the order and its receipt into inventory) or the time required for acquiring an item is also known.

Although above two assumptions are rarely valid for inventory problems in the business world, they do allow us to develop a simple model into which more realistic, complicating factors can be introduced.

Let q be the order size under the preceding assumptions, it can be shown from the Fig. 20·2. that the number of units in inventory is equal to q when each new order is practically received into inventory and that the inventory is gradually decreased until it becomes zero just at the point when the next order is received.

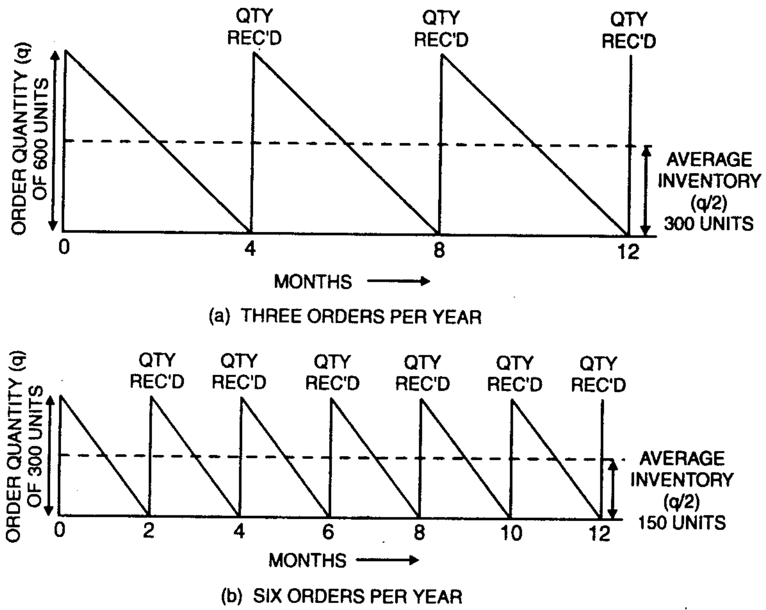


Fig. 20.2. (*a*) Three orders per year, (*b*) Six orders per year.

Also, it is observed that the average inventory ($q/2$) is equal to one-half the number of units in the lot size. As shown in the Fig. 20·2, the average inventory is affected by the order quantity and the number of orders per year. Moreover, each new order is received into inventory at exactly the time at which the previous order is decreased, resulting in no stockouts.

20.12. CONCEPT OF ECONOMIC ORDERING QUANTITY (EOQ)

The concept of *economic ordering quantity* was first developed by *F. Harris* in 1916. The concept is that management is confronted with a set of opposing costs—as the lot size (q) increases, the carrying charges (C_1) will increase while the ordering costs (C_3) will decrease. On the other hand, as the lot size (q) decreases, the carrying cost (C_1) will decrease but the ordering costs will increase (assuming that only minor deviations from these trends may occur). Thus, ***economic ordering quantity (EOQ) is that size of order which minimizes total annual (or other time period as determined by individual firms) cost of carrying inventory and cost of ordering under the assumed conditions of certainty and that annual demands are known.***

The concept of EOQ applies to items which are replenished periodically into inventory in lots covering several period's need. The EOQ concept is applicable under the following conditions :

(a) The item is replenished in lots or batches, either by purchasing or by manufacturing.

(b) Consumption of items (or sales or usage rate) is uniform and continuous.

EOQ is that order quantity or optimal order size which minimises the total cost.

The model is described under the following situations :

(i) Planning period is one year.

(ii) Demand is deterministic and indicated by parameter D units per year.

(iii) Cost of purchases, or of one unit is C.

(iv) Cost of ordering (or procurement cost of replenishment cost) is C_3 or C_o. For manufacturing goods, it is known as set-up cost.

(v) Cost of holding stock (also known as inventory carrying cost) is C_1 or C_h per unit per year experssed either in items of cost per unit per period or in terms of percentage charge of the purchase price.

(vi) Shortage cost (mostly it is back order cost) is C_2 or C_s per unit per year.

(vii) Lead time is L, expressed in unit of time.

(viii) Cycle period in replenishment is t.

(ix) Order size is Q.

Q. 1. What are the basic ideas involved in EOQ concept ? Discuss. **[IGNOU 99, 98, 97, 96]**

2. What is Economic Order Quantity? Discuss step by step the development of EOQ formula. **[Bhubnaswar (IT) 2004]**

3. It is said that 'EOQ models, however complex, are restricted by so many assumptions that they have very limited practical value'. Do you agree with this view ? Illustrate your answer with examples.

20.12–1. Determination of EOQ by Trial and Error Method (or Tabular Method)

This method involves the following steps :

***Step* 1.** Select a number of possible lot sizes to purchase.

***Step* 2.** Determine total cost for each lot size chosen.

***Step* 3.** Finally, select the ordering quantity which minimizes total cost.

Following illustrative example will make the procedure clear.

For example, suppose annual demand (D) equals 8000 units, ordering cost (C_3) per order is Rs. 12·50, the carrying cost of average inventory is 20% per year, and the cost per unit is Re. 1·00. The following table is computed.

No. of Orders per Year	Lot Size	Average Inventory	Carrying Charges 20% per Year	Ordering Costs Rs. 12·50 per Order	Total Cost per Year
(1)	(2)	(3)	(4)	(5)	(6) = (4) + (5)
			(Rs.)	(Rs.)	(Rs.)
1	8,000	4,000	800·00	12·50	812·50
2	4,000	2,000	400·00	25·00	425·00
4	2,000	1,000	200·00	50·00	250·00
→ 8	1,000	500	100·00	100·00	200·00
12	667	333	66·00	150·00	216·00
16	500	250	50·00	200·00	250·00
32	50	125	25·00	400·00	425·00

This table indicates that an order size of 1000 units will give us the lowest total cost among all the seven alternatives calculated in the table. Also, it is important to note that this minimum total cost occurs when :

$$\boxed{\textbf{Carrying Costs} = \textbf{Ordering Costs.}}$$

Example 1. *Novelty Ltd. carries a wide assortment of items for its customers. One item, Gaylook, is very popular. Desirous of keeping its inventory under control, a decision is taken to order only the optimal* *economic quantity, for this item, each time. You have the following information. Make your recommendations :*

Annual Demand : 1,60,000 *units; Price per unit : Rs.* 20; *Carrying Cost : Re.* 1 *per unit or 5% per rupee of inventory value; Cost per order : Rs.* 50.

Determine the optimal economic quantity by developing the following table.

No. of Orders	Size of Orders	Average Inventory	Carrying Costs	Ordering Costs	Total Cost
1	...	...	...	...	...
10	...	...	...	...	...
20	...	...	...	...	...
40	...	...	...	...	...
80	...	...	...	...	...
100	...	...	...	...	...

Solution.

Tabular Method to Find EOQ

Orders per Year	Lot Size	Average Inventory	Carrying Cost (Re. 1) (Rs.)	Ordering Cost (Rs. 50 per order) (Rs.)	Total Cost per Year (Rs.)
1	1,60,000	80,000	80,000	50	80,050
10	16,000	8,000	8,000	500	8,500
20	8,000	4,000	4,000	1,000	5,000
40	4,000	2,000	2,000	2,000	4,000
80	2,000	1,000	1,000	4,000	5,000
100	1,600	800	800	5,000	5,800

Disadvantage of trial & error (or tabular) method.

In above example, we were fortunate enough in finding the lowest possible cost. But, suppose the computation for 8 orders per year had not been made.

Then, we could choose only among the six remaining alternatives for the lowest cost solution. This imposes a serious limitation of this method. So, a relatively large number of alternatives must be computed before the best possible least cost solution is obtained. In this situation, the following *graphical method* may be advantageous.

20.12-2. Graphical Method

The data calculated in the above table can be graphed as below to demonstrate the nature of the opposing costs involved in an EOQ model.

This graph shows that annual total costs of inventory, carrying costs and ordering costs first decrease, then hit a lowest point where *inventory carrying costs* equal *ordering costs*, and finally increases as the ordering quantity increases. Our main objective is to find a numerical value for EOQ that will minimize the total variable costs on the graph.

Disadvantage of graphical Method. Without specific costs and values an accurate plotting of the carrying costs, ordering costs, and total costs is not feasible.

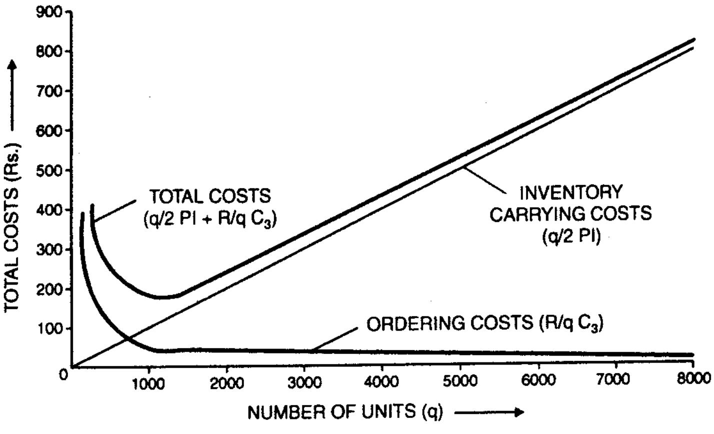


Fig. 20.3 Economic ordering quantity graph

In this chapter, we shall solve the EOQ models by the following two more accurate methods.

Algebraic method. This method is based on the fact that the most economical point in terms of total inventory cost is where the *inventory carrying cost* equals *ordering cost.*

Calculus method. This method is based on the technique of finding the minimum total cost by utilizing the differentiation. This is the best method since it does not suffer from the limitations like previous methods.

I– Static Demand Models (I, II, III)

Under this section, we shall deal with the inventory models in which demand is assumed to be fixed and completely pre-determined, *i.e.*, ***static demand.*** Such models are usually referred to as ***Economic lot size models.*** We now proceed to discuss **Model I, II** and **III** as classified in **Section 20.10.** We have further classified each model into three sub-classes [**(a)**, **(b)**, **(c)**] in which the assumptions are slightly changed.

If the known demand is ***dynamic*** *(i.e.,* ***fluctuating****)*, then such models are classified as ***Model IV*** and ***Model V*** with slight variation in their assumptions.

20.13. THE EOQ MODEL WITHOUT SHORTAGE

20.13-1. Model I (a) : The Economic Lot Size System with Uniform Demand

In this model, we want to derive an economic lot size formula for the optimum production quantity q per cycle (i.e. per production run) of a single product so as to minimize the total average variable cost per unit time, where

(i) demand is uniform at a rate of R quantity units per unit time,

(ii) lead time is zero (or known exactly),

(iii) production rate is infinite, i.e. production is instantaneous,

(iv) shortages are not allowed,

(v) holding cost is rupees C_1 per quantity unit per unit time,

(vi) set-up cost is rupees C_3 per set-up. **[Meerut (Maths.), 2002, 2000; Jan. 98 BP; Rohilkhand 95]**

Method I —Algebraic Method

As explained earlier, the most economic point in terms of total inventory cost exists where,

Inventory carrying costs = Annual ordering (set-up) costs. ...(20.1)

This is the principal basis of algebraic method.

Since the demand is uniform and known exactly and supply is instantaneous, the recorder point is that when inventory falls to zero. Rise and fall in inventory level for a particular item over time reflects the periodic cycles of depletion and replenishment as shown in the Fig. 20·4. Since the actual consumption of inventory varies constantly, the concept of average inventory is applicable here. With a constant rate of demand,

Average inventory = ½ [maximum level + minimum level] = $(q + 0)/2 = q/2$.

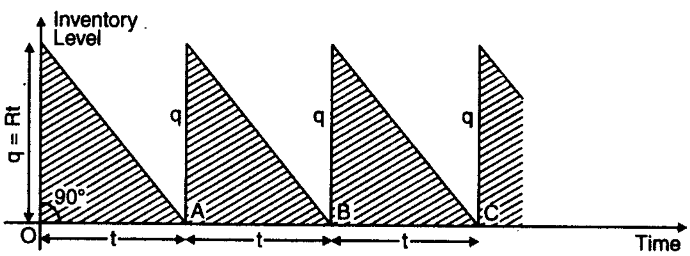


Fig. 20.4.

Total inventory carrying costs are determined in the following manner :

$$\begin{pmatrix}\text{average no.}\\ \text{of units in}\\ \text{inventory}\end{pmatrix} \times \begin{pmatrix}\text{cost of}\\ \text{one}\\ \text{unit}\end{pmatrix} \times \begin{pmatrix}\text{inventory}\\ \text{carrying cost}\\ \text{percentage}\end{pmatrix} = \begin{pmatrix}\text{total inventory}\\ \text{carrying costs}\\ \text{per unit}\end{pmatrix}$$

or $$\tfrac{1}{2} q \times C \times I = \tfrac{1}{2} qCI = \tfrac{1}{2} qC_1 \qquad ...(20.2)$$

where the term (CI) can be written simply as C_1 (the holding or carrying cost per unit for a unit time).

Total annual ordering costs are obtained as follows :

$$\begin{pmatrix}\text{number of orders}\\ \text{per year}\end{pmatrix}\times\begin{pmatrix}\text{ordering cost}\\ \text{per order}\end{pmatrix}=\begin{pmatrix}\text{total ordering}\\ \text{costs}\end{pmatrix}$$

or $$(R/q) \quad \times \quad C_3 \quad = \quad (R/q)\,C_3 \qquad ...(20.3)$$

Now summing-up the total inventory carrying cost and total ordering cost, we get the total inventory cost,

$$\begin{pmatrix}\text{total inventory}\\ \text{costs}\end{pmatrix}=\begin{pmatrix}\text{total inventory}\\ \text{carrying costs}\end{pmatrix}+\begin{pmatrix}\text{total annual}\\ \text{ordering costs}\end{pmatrix} \quad \textbf{(Cost Equation)}$$

or $$C(q) \quad = \quad ½\,qC_1 \quad + \quad (R/q)\,C_3 \qquad ...(20.4)$$

But, the total inventory cost $C(q)$ is minimum when the inventory carrying costs become equal to the total annual ordering costs. Therefore,

$$\frac{1}{2}qC_1=\frac{R}{q}C_3 \text{ or } qC_1=\frac{2R}{q}C_3 \text{ or } q^2=\frac{2C_3R}{C_1} \text{ or } q=\sqrt{\frac{2C_3R}{C_1}} \qquad ...(20.5.)$$

$$\boxed{\textbf{Optimal } q^* \textbf{ (EOQ)} = \sqrt{\left(\frac{\textbf{2} \times \textbf{setup cost} \times \textbf{demand rate}}{\textbf{carrying cost}}\right)}}$$

Now to obtain the minimum of total inventory cost $C(q)$, we substitute the value of q from eqn. (20·5) in the cost equation (20·4), and get

$$C_{\min}=\frac{1}{2}\sqrt{\frac{2C_3R}{C_1}}\times C_1+\frac{RC_3}{\sqrt{2C_3R/C_1}}=\frac{1}{2}\sqrt{2C_1C_3R}+RC_3\times\sqrt{C_1/2C_3R}$$

$$=\sqrt{\frac{2C_1C_3R}{2^2}}+\sqrt{\frac{C_1R^2C_3^2}{2C_3R}}=\sqrt{\frac{C_1C_3R}{2}}+\sqrt{\frac{C_1C_3R}{2}}$$

$$=2\sqrt{\frac{C_1C_3R}{2}}=\sqrt{2C_1C_3R} \qquad \textbf{(Minimum Cost)}$$

Hence, $C_{\min}=\sqrt{2C_1C_3R}$, *i.e.* ...(20.6)

$$\boxed{\textbf{Optimum inventory cost } (C_{\min}) = \sqrt{[\textbf{2} \times \textbf{(holding cost)} \times \textbf{(setup cost)} \times \textbf{(demand rate)}]}}$$

To obtain the optimum interval of ordering (t^*), we have

$$\begin{pmatrix}\text{economic ordering}\\ \text{quantity}\end{pmatrix}=\begin{pmatrix}\text{demand}\\ \text{rate}\end{pmatrix}\times\begin{pmatrix}\text{interval of}\\ \text{ordering}\end{pmatrix}$$

or $$q \quad = \quad R \quad \times \quad t$$

or $$t=\frac{q}{R}=\frac{1}{R}\times\sqrt{\frac{2C_3R}{C_1}}=\sqrt{\frac{2C_3}{RC_1}}$$

Hence $t=\sqrt{\dfrac{2C_3}{RC_1}}$, *i.e.* ...(20.7)

$$\boxed{\textbf{Optimum ordering interval } (t^*) = \sqrt{\frac{\textbf{2} \times \textbf{setup cost}}{\textbf{demand rate} \times \textbf{holding cost}}}}$$

We determine the optimum number of orders to be placed by the formula $N = (R/q)$, *i.e.*

$$\boxed{\textbf{Optimum number of orders } (N) = \frac{\textbf{total annual quantity requirement } (R)}{\textbf{economic ordering quantity } (q)}} \qquad ...(20.8)$$

The number of days' supply per optimum order is obtained by the formula, $d = 365/N$, *i.e.*

$$\boxed{\textbf{Number of days' supply } (d) = \frac{\textbf{365}}{\textbf{optimum number of orders } (N)}}$$

To illustrate the *EOQ* model numerically, we use the example of graphical method (Sec **20.12–1,** page 687), where

C = Re. 1·00 , I = 20%, R = 8000 units, and C_3 = Rs. 12·50. Then,

$$q = \sqrt{\frac{2 \times 8000 \times (\text{Rs. } 12{\cdot}50)}{(\text{Re. } 1{\cdot}00) \times (20\%)}} = \sqrt{\frac{16000 \times 12{\cdot}50}{0{\cdot}20}} = \sqrt{\frac{2{,}00{,}000}{0{\cdot}20}} = \sqrt{(1{,}000{,}000)} = 1000 \text{ units.}$$

Substituting the value for q in the original terms of the model, we obtain :

(i) Total inventory carrying cost = $(q/2)\, CI = (½ \times 1000) \times$ (Re. 1) $\times$ 20% = Rs. 100.

(ii) Total ordering cost = $(R/q)\, C_3 = \frac{8000}{1000} \times$ (Rs. 12·50) = Rs. 100.

These costs can be compared with those obtained by graphical method.

The adding of above two costs equals the lowest (minimum) cost per year of Rs. 200 for the economic ordering quantity, which can also be obtained by substituting the relevant quantities in formula (20·6), *i.e.*

$$C_{\min} = \sqrt{2 \times (Re.\ 1{\cdot}00 \times 20\%) \times (Rs.\ 12{\cdot}50) \times 8000} = \sqrt{2 \times 0{\cdot}20 \times 12{\cdot}50 \times 8000} = \text{Rs. } 200.$$

This example illustrates that we have solved the model for minimum cost.

Optimum number of orders are obtained by :

$$N = \frac{\text{total annual requirement}}{\text{economic ordering quantity}} = \frac{8000}{1000} = 8 \text{ orders per year.}$$

The number of days' supply per optimum order is obtained by

$$d = \frac{365}{\text{optimum number of orders}} = \frac{365}{8} = 45{\cdot}8 \text{ days.}$$

Method II—Calculus Method

Let each production cycle be made at fixed interval t and, therefore, the quantity q already present in the beginning (when the business was started) should be

$$q = Rt \text{ , where } R \text{ is the demand rate.} \qquad ...(20.9)$$

Since the stock in small time dt is $Rt\, dt$, the stock in total time t will be

$$= \int_0^t Rt\, dt = ½\, Rt^2 = ½\, qt = \text{area of the inventory } \Delta POA \text{ [since } Rt = q \text{ in (20.9)]}$$

The graphical representation of this inventory problem is already shown in Fig. 20·4, page 689.

Clearly, the rate of replenishment = slope of line OP = tan 90° = ∞

Thus, the cost of holding inventory per production run = $C_1 \times$ Area of $\Delta OPA = C_1\, (½\, Rt^2)$, ...(20.10)

and the set-up cost (*i.e.* production cost) = C_3 per production run for interval t. ...(20.11)

Hence, summing up the costs in (20.10) and (20.11) and dividing by t, we get the average total cost given by

$$C(t) = ½\, C_1 Rt + C_3/t \quad \textit{(Cost Equation)} \qquad ...(20.12)$$

The condition for min. or max. of $C(t)$ is $\frac{dC(t)}{dt} = 0$.

Hence, on differentiating $C(t)$ in eqn. (20.12) and then equating it to zero, we get

$$½\, C_1 R - C_3/t^2 = 0,$$

which gives us

$$t = \sqrt{\frac{2C_3}{C_1 R}}. \qquad ..(20.13)$$

Also, on differentiating $C(t)$ in (20.12) twice, we get

$$\frac{d^2 C(t)}{dt^2} = 0 + \frac{2C_3}{t^3}$$

which is obviously positive for the value of t given by eqn. (20.13).

Hence, $C(t)$ is minimum for optimum time interval

$$t^* = \sqrt{\frac{2C_3}{C_1 R}} \qquad ...(20.14)$$

and optimum quantity to be produced (or ordered) at each interval t^* is given by

$$q^* = Rt^* = R\sqrt{\frac{2C_3}{C_1 R}} = \sqrt{\frac{2C_3 R}{C_1}} \qquad ...(20.15)$$

which is called the ***optimal lot size formula.***

This will result in minimum cost from eqn. (20.12) of the value

$$c_{\min} = \tfrac{1}{2} RC_1 \sqrt{\frac{2C_3}{C_1 R}} + C_3 \sqrt{\frac{C_1 R}{2C_3}} = \sqrt{\frac{C_1 C_3 R}{2}} + \sqrt{\frac{C_1 C_3 R}{2}} = \sqrt{2C_1 C_3 R} \text{ per unit time} \quad \ldots(20.16)$$

Note. The cost equation (20.12) can also be written in an alternative form on replacing t by q/R as :

$$C(q) = \tfrac{1}{2} C_1 q + C_3 R/q \quad \ldots(20.17)$$

Q. 1. What are the assumptions of Economic lot size formula.

2. Write short note on Economic lot size problem with uniform rate of demand.

3. Derive an Economic lot size formula for the optimum production quantity q per cycle so as to minimize the total average cost per unit time, where lead time is zero, demand is uniform, production is instantaneous and there are no shortages.

4. Derive economic order quantity model for an inventory problem when shortages of costs are not allowed. **[Garhwal M.Sc (Math.) 92]**

5. Describe the single item production inventory model with no shortages and derive the formulae for optimum lot size for one run and the optimum time between two runs. **[Virbhadrah 2000]**

20.13–2. Model I (*b*) : Economic lot size with different rates of demand in different cycles

If in the **Model I(*a*)**, the total demand D is prescribed over the total time period T instead of demand rate being constant for each production cycle, *i.e.*, rate of demand being different in different production cycles, then derive the optimal lot size formula and the minimum cost.

Method I—Algebraic Method

Let the total demand D be specified as demand during total time period T and q be the stock level to be fixed. Thus the inventory costs are determined as follows :

$$\text{Ordering costs} = (\text{Number of orders}) \times C_3 = \left(\frac{D}{q}\right) C_3 \quad \ldots(20.18)$$

$$\text{Carrying costs} = (\text{Average inventory}) \times C_1 \times T = \left(\frac{q}{2}\right) C_1 T. \quad \ldots(20.19)$$

The total inventory cost is the sum of above two cases, given by

$$C(q) = \left(\frac{q}{2}\right) C_1 T + \left(\frac{D}{q}\right) C_3 \quad \textbf{(Cost Equation)} \quad \ldots(20.20)$$

As discussed in ***Model I (a)***, the optimal ordering quantity (q^*) is determined by equating the ordering costs and carrying costs, *i.e.*,

$$\left(\frac{q}{2}\right) C_1 T = \left(\frac{D}{q}\right) C_3 \text{ or } q^2 = \frac{2DC_3}{C_1 T} \text{ or } q = \sqrt{\frac{2C_3 (D/T)}{C_1}} \quad \ldots(20.21)$$

The minimum total yearly inventory cost is obtained by substituting the value of q from (20.21) in the cost equation (20.20), *i.e.*,

$$C_{\min} = 1/2 C_1 T \sqrt{\frac{2C_3 D}{C_1 T}} + DC_3 \sqrt{\frac{C_1 T}{2C_3 D}} = \sqrt{2C_1 C_3 (D/T)} \quad \ldots(20.22)$$

Method II—Calculus Method

Let q be the fixed quantity produced in each production cycle.

Since D is the total demand prescribed over the time period T, the number n of production cycles will be given by $n = D/q$

Also, let the total period $T = t_1 + t_2 + t_3 + \ldots + t_n$.

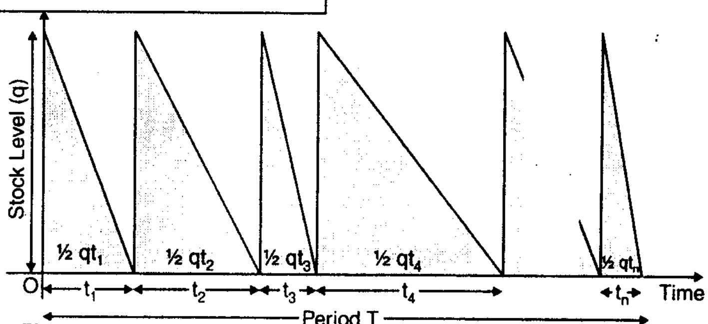


Fig. 20.5. Diagram of inventory level with different rates of demand in different cycles.

It is obvious that the fixed quantity q, produced in the beginning of interval t_1, is supposed with uniform rate of demand in interval t_1 only.

(**Note :** Similarly, the same quantity q is again produced in the beginning of next inverval t_2 which is supplied with some other different uniform rate of demand during this interval t_2, and so on for the remaining intervals $t_3, t_4, t_5, \ldots, t_n$).

Graphical representation of this model is given in Fig. 20.5. :

Obviously, the carrying cost for the period T will be

$$= (\tfrac{1}{2} qt_1) C_1 + (\tfrac{1}{2} qt_2) C_1 + \ldots + (\tfrac{1}{2} qt_n) C_1$$
$$= \tfrac{1}{2} C_1 q (t_1 + t_2 + t_3 + \ldots + t_n) = \tfrac{1}{2} C_1 qT,$$

and the set-up cost will be $= \left(\frac{D}{q}\right) C_3$ (because C_3 is the set-up cost per cycle)

Thus, we obtain the cost equation for period T as

$$C(q) = \tfrac{1}{2} C_1 qT + \frac{D}{q} C_3 \quad \textbf{(Cost Equation)} \qquad \ldots(20.23)$$

For optimum cost $$\frac{dC}{dq} = \tfrac{1}{2} C_1 T - \frac{C_3}{q^2} D = 0 \qquad \ldots(20.24)$$

which gives $$q = \sqrt{\frac{2C_3 (D/T)}{C_1}}$$

Also, $\frac{d^2C}{dq^2} = \frac{2C_3 D}{q^3}$, which is positive. Therefore, the optimum lot size is given by

$$q^* = \sqrt{\frac{2C_3 (D/T)}{C_1}} \qquad \ldots(20.25)$$

which minimizes the total cost $C(q)$ given by eqn. (20.23).

Substituting the value of q^* in (20.23), we get

$$C_{\min} = \tfrac{1}{2} C_1 T \sqrt{\frac{2C_3 (D/T)}{C_1}} + C_3 D \sqrt{\frac{C_1}{2C_3 (D/T)}}$$
$$= \sqrt{\frac{C_1 C_3 TD}{2}} + \sqrt{\frac{C_1 C_3 TD}{2}} = \sqrt{2C_1 C_3 DT}.$$

Hence the minimum average cost will be

$$C_{\min} = \sqrt{(2C_1 C_2 DT)}/T = \sqrt{2C_1 C_3 (D/T)}. \qquad \ldots(20.26)$$

Here we observe that the 'fixed' demand rate R in ***Model I (a)*** is replaced by the average demand rate (D/T) in this model. In other words, replacing R by D/T in the results (20.14), (20.15) and (20.16) of ***Model I (a)***, we can easily obtain the corresponding results for ***Model I (b)***.

Q. 1. Prove that in inventory problem of Economic lot size with uniform demand and unequal times of production run, the optimal lot size Q for each production run is given by $Q^* = \sqrt{2D' C_3/C_1}$ and the optimal total cost C^* is given by $C^* = \sqrt{2D' C_1 C_3}$ where D' denotes the total number of units produced per unit time, C_3 is the set-up cost per production run and C_1 is the holding cost per unit of inventory per unit time. Production is instantaneous and shortage cost infinite. [**Hint.** $D' = D/T$ in Model I (b)]

2. Derive an economic lot size with different rates of demand in different cycles. **[Garhwal M.Sc (Math) 94]**

Illustrative Examples

Example 2. *Find the EOQ for the folowing data :*

Annual usage = 1,000 pieces	*Expediting cost* = Rs. 4 per order
Cost per piece = Rs. 250	*Inventory holding cost* = 20% of average inventory
Ordering cost = Rs. 6 per order	*Material holding cost* = Re. 1 per piece.

[Shivaji (MBA) 96]

Solution. Here $D = 1{,}000$ pieces

$C_3 = \text{Rs. } 6 + \text{Rs. } 4 = \text{Rs. } 10$ per order

$C_1 = C \times I = 250 \times 0{\cdot}20$

$$\therefore \quad Q^* (EOQ) = \sqrt{\frac{2 \times 10 \times 1000}{250 \times 0{\cdot}20}} = 20.$$

Remark. The material handling cost is ignored since it remains the same whatever be the batch size.

Example 3. *You have to supply your customers 100 units of a certain product every Monday (and only then). You obtain the product from a local supplier at Rs. 60 per unit. The costs of ordering and transportation from the supplier are Rs. 150 per order. The cost of carrying inventory is estimated at 15% per year of the cost of the product carried.*

(i) Describe graphically the inventory system.

(ii) Find the lot size which will minimize the cost of the system.

(iii) Determine the optimal cost. **[Rohilkhand 95]**

Solution. In this problem, $R = 100$ units/week $C_3 = \text{Rs. } 150$ per order

$C_1 = 15\%$ per year of the cost of the product carried. = Rs. $(15 \times 60)/(100 \times 52)$ per unit per week.

= Rs. $9/52$ per unit per week. (1 year = 52 weeks)

(i) For graphical description see Fig. 20.4, Page 677.

(ii) $q^* = \sqrt{\dfrac{2C_3R}{C_1}} = \sqrt{\dfrac{2 \times 150 \times 100 \times 52}{9}} = 416$ units.

(iii) $C_{\min} = 60R + \sqrt{(2C_1C_3R)} = 6000 + \sqrt{2 \times (9/52) \times 150 \times 100} = \text{Rs. } 6072.$

Note. The students are advised to solve this problem independently by forming cost equation :

$$C(t) = \frac{1}{2} RC_1t + 60R + \frac{C_3}{t}.$$

Example 4. *If in **Model I(a)**, the set-up cost instead of being fixed is equal to $(C_3 + Bq)$, where B is the set-up cost per unit item produced; then show that there is no change in the optimum order quantity produced due to this change in the set-up cost.*

Solution. Now substituting the set-up cost $(C_3 + Bq)$, instead of C_3, in the cost equation (20.12), we get the new cost equation

$$C(t) = \tfrac{1}{2} C_1 Rt + (C_3 + Bq)/t.$$

or $$C(t) = \tfrac{1}{2} C_1 Rt + (C_3 + BRt)/t. \qquad (\because q = Rt)$$

or $$C(t) = \frac{1}{2} C_1 Rt + \frac{C_3}{t} + BR.$$

For optimum cost $$\frac{dC}{dt} = \frac{1}{2} C_1 R - \frac{C_3}{t^2} + 0 = 0.$$

which gives $$t^* = \sqrt{\frac{2C_3}{C_1R}}, \text{ and } q^* = Rt^* = \sqrt{\frac{2C_3R}{C_1}},$$

which is the same as eqn. (20.15).

Thus, there will be no change in the optimal order quantity produced due to this change in set-up cost.

Example 5. *An aircraft company uses rivets at an approximate customer rate of 2,500 kg. per year. Each unit costs Rs. 30 per kg. and the company personnel estimate that it costs Rs. 130 to place an order, and that the carrying cost of inventory is 10% per year. How frequently should orders for revets be placed ? Also determine the optimum size of each order.*

Solution. Here we have, $R = 2{,}500$ per year,

C_1 = (cost of each unit) × (inventory carrying cost). = Rs. 30×0.1 per unit per year

$$\therefore \quad q^* = \sqrt{\frac{2RC_3}{C_1}} = \sqrt{\frac{2 \times 2500 \times 130}{30 \times 0.1}} = 466 \text{ units (approx.)}$$

$t^* = q^*/R = 466/2500 = 0.18$ year $= 2.16$ months.

n = number of orders $= R/q^* \simeq 5$ orders per year.

Example 6. *An aircraft uses rivets at an approximately constant rate of 5,000 kg. per year. The rivets cost Rs. 20 per kg. and the company personnel estimate that it costs Rs. 200 to place an order, and the carrying cost of inventory is 10% per year.*

(i) How frequently should orders for rivets be placed, and what quantities should be ordered for ?

(ii) If the actual costs are Rs. 500 to place an order and 15% for carrying cost, the optimum policy would change. How much is the company losing per year because of imperfect cost information ?

[AIMA (PG. Dip. in Management) Dec. 95]

Solution. In usual notations, we are given :

Demand rate (D) = 5,000 kg per year

Cost (C) = Rs. 20 per kg.

Inventory carrying rate (I) = 10% per year

Inventory carrying cost, $C_h = I \times C = 20 \times 0{\cdot}1$ = Rs. 2 per year

ordering cost, C_0 = Rs. 200 per order

$$\text{(i)}\quad Q^*(EOQ) = \sqrt{\frac{2DC_0}{C_h}} = \sqrt{\frac{2 \times 5{,}000 \times 200}{2}} = 1{,}000 \text{ units.}$$

and, optimum number of orders per year,

$$N^* = \frac{D}{Q^*} = \frac{5{,}000}{1{,}000} = 5 \text{ orders}$$

Total minimum variable inventory cost corresponding to Q^* = 1000 units

$$TVC^* = \sqrt{2DC_0C_h} = \sqrt{2 \times 5{,}000 \times 200 \times 2} = 2 \times 5{,}000 \times 200 \times 2 = \text{Rs. } 2{,}000.$$

(ii) If C_0 = Rs. 500 per order and $I = 0{\cdot}15$, then

$$Q^* = \sqrt{\frac{2 \times 5{,}000 \times 500}{20 \times 0{\cdot}15}} = 1{,}291 \text{ units.}$$

and $\quad TVC^* = \sqrt{2 \times 5{,}000 \times 500 \times 3} = 3{,}873.$

The loss per year due to imperfect information = (3,873 − 2,000) = Rs. 1,873.

Example 7. *A manufacturing company purchases 9,000 parts of a machine for its annual requirements, ordering one month usage at a time. Each part costs Rs. 20. The ordering cost per order is Rs. 15, and the carrying charges are 15% of the average inventory per year. You have been asked to suggest a more economical purchasing policy for the company. What advice would you offer, and how much would it save the company per year ?* **[JNTU 99]**

Solution. Here we have, R = 9,000 parts per year, C_3 = Rs. 15 per order

C_1 = 15% of the average inventory per year $= 20 \times (15/100)$ = Rs. 3 each part per year

$$\therefore\quad q^* = \sqrt{\frac{2C_3R}{C_1}} = \sqrt{\frac{2 \times 15 \times 9{,}000}{3}} = 300 \text{ units.}$$

$t^* = q^*/R = 300/9{,}000 = 1/30$ year $= 365/30 = 12$ days.

$C_{\min} = \sqrt{2C_1C_3R} = \sqrt{2 \times 3 \times 15 \times 9{,}000}$ = Rs. 900.

If the company follows the policy of ordering every month, then the annual ordering cost becomes

$= 12 \times 15$ = Rs. 180,

and lot size of inventory each month, $q = 9000/12 = 750$ parts.

Average inventory at any time $= \frac{1}{2}q = 750/2 = 375$ parts.

Storage cost at any time $= 375\ C_1 = 375 \times 3$ = Rs. 1,125.

Total annual cost = 1,125 + 180 = Rs. 1,305.

Therefore, the company purchases 300 parts at time intervals of 12 days instead of ordering 750 parts each month. So there will be a net saving of Rs. 1,305 − Rs. 900 = Rs. 405 per year **Ans.**

Example 8. *Data relevant to component R used by Engineering India Private Limited in 20 different assemblies includes—Purchase price : Rs. 15 per 100, annual usage : 100,000 units, cost of buying office :*

fixed Rs. 15,575 per annum, variable cost : Rs. 12 per order, rent of component : Rs. 3000 per annum, heating : Rs. 700 per annum, interest : 25% per annum, insurance 0.05% per annum based on total purchases, depreciation as 1% per annum of all items purchased. Calculate :

(i) EOQ *for component R.*

(ii) The percentage changes in total annual variable costs relating to component R if the annual usage was (a) 125,000 units and (b) 75,000 units. **[Allahabad (M.B.A.) 90]**

Solution. In usual notations, we are given that

$$R = 100{,}000,\ C_3 = 12,\ C_1 = (15/100) \times (0.25 + 0.0005 + 0.01) = 0.039075.$$

The necessary calculations may be presented in the following tabular form :

Demand per year	$q^* = \sqrt{(2C_3R)/C_1}$	Ordering cost (Rs.)	Holding cost (Rs.)	Toal annual variable cost (Rs.)
100,000	$\sqrt{\dfrac{2 \times 12 \times 100{,}000}{0.039075}}$ = 7,837 units	$\dfrac{100{,}000}{7{,}837} \times 12$ = 171.12	$\dfrac{7{,}837}{2} \times .039075$ = 153.12	306.25
125,000	$\sqrt{\dfrac{2 \times 12 \times 125{,}000}{0.039075}}$ = 8,762	$\dfrac{125{,}000}{8{,}762} \times 12$ = 171.12	$\dfrac{8{,}762}{2} \times .039075$ = 171.19	342.31
75,000	$\sqrt{\dfrac{2 \times 12 \times 75{,}000}{0.039075}}$ = 6,787	$\dfrac{75{,}000}{6{,}787} \times 12$ = 132.6	$\dfrac{6{,}787}{2} \times .039075$ = 132.6	265.20

Therefore, when the annual demand was 125,000 units the total annual variable cost has increased by approx. 12%, whereas it has decreased by 13% for annual demand 75,000.

Example 9. *A contractor has to supply 10,000 bearings per day to an automobile manufacturer. He finds that, when he starts production run, he can produce 25,000 bearings per day. The cost of holding a bearing in stock for a year is Rs. 2 and the set up cost of a production run is Rs. 1,800. How frequently should production run be made ?* **[AIMA (P.G. Dip. In Management) May 96]**

Solution. In the usual notations, we are given :

Annual demand, D = 10,000 × 300 = 30,00,000 units (Assuming 300 working days in the year)

Set up cost, C_0 = Rs. 1,800 per run

Demand rate, d = 10,000 bearings per day

Production rate, p = 25,000 bearings per day

Holding cost, C_h = Rs. 2 per year

(i) Optimum order quantity, Q^* for each production run is :

$$Q^* = \sqrt{\left[\frac{2DC_0}{C_h} \times \frac{1}{[1-(d/p)]}\right]}$$

$$= \sqrt{\left[\frac{2 \times 30{,}00{,}000 \times 1{,}800}{2} \times \frac{25{,}000}{(25{,}000 - 10{,}000)}\right]}$$

= 104,446 bearings.

(ii) Frequency of production runs is given by

$$T^* = \frac{Q^*}{d} = \frac{104{,}446}{10{,}000} = 10{\cdot}44 \text{ days}$$

Thus, the production run can be made after every 10·44 days.

Example 10. *The details of a part to be machined are as follows :*

Annual requirement = 2400 pieces, *Machine rate* = 10 pieces/shift

No. of working days in the year = 320 shifts

Cost of machining a component = Rs. 100 per piece.

Inventory carrying cost per annum = 12% of value

Set up cost per production run = Rs. 400

Find the optimum run size for machining. **[Madras (MBA) 97]**

Solution. In the usual notations, we are given :

D = 2,400 pieces per year

C_0 = Rs. 400 per production run

p = 320 × 10 or 3200 pieces per year (production rate)

$$C_h = CI = 100 \times \frac{12}{100} = \text{Rs. } 12 \text{ per year.}$$

Economic lost size for each production run is :

$$Q^* = \sqrt{\frac{2DC_0}{C_h}\left(\frac{p}{p-D}\right)}$$

$$= \sqrt{\frac{2 \times 2400 \times 400}{12}\left(\frac{3200}{3200-2400}\right)} = 800 \text{ pieces}$$

Example 11. *A manufacturing company uses an EOQ (Economic order quantity) approach in planning its production of gears. The following information is available. Each gear costs Rs. 250 per unit, annual demand is 60,000 gears, set up costs are Rs. 4,000 per set up and the inventory carrying cost per month is established at 2 per cent of the average inventory value. When in production, these gears can be produced at the rate of 400 units per day and this company works only for 300 days in a year. Determine the economic lot size, the number of production runs per year and the total inventory costs.* **[Shivaji (MBA) Nov. 98]**

Solution. In the usual notations, we are given

D = 60,000 gears per year, C = Rs. 250 per gear

C_0 = Rs. 4,000 per set up, C_h = 24% of Rs. 250 = Rs. 60 per year.

$$d = \frac{60,000}{300} = 200 \text{ gears per day } (\textit{replenishment rate})$$

p = 400 gears per day (*replenishment rate*)

(i) Economic lot size for each production run is given by :

$$Q^* = \sqrt{\frac{2DC_0}{C_h}\left(\frac{p}{p-D}\right)}$$

$$= \sqrt{\frac{2 \times 60,000 \times 4,000}{60}\left(\frac{400}{400-200}\right)} = 4,000 \text{ gears}$$

(ii) The number of production runs per year is given by :

$$N^* = \frac{D}{Q^*} = \frac{60,000}{4,000} = 15 \text{ production runs per annum.}$$

(iii) The total inventory cost is given by :

$$TC(Q^*) = C_0 \times \frac{D}{Q^*} + C_h \times \frac{Q^*}{2} \times \left(1 - \frac{d}{p}\right)$$

$$= \text{Rs. } 4,000 \times \frac{60,000}{4,000} + \text{Rs. } 60 \times \frac{4,000}{2}\left(1 - \frac{200}{400}\right) = \text{Rs. } 1,20,000.$$

Example 12. *A dealer supplies you the following information with regard to a product dealt in by him :*

Annual demand : 10,000 units; ordering cost : Rs. 10 per order; price : Rs. 20 per unit.

Inventory carrying cost : 20% of the value of inventory per year.

The dealer is considering the possibility of allowing some back order (stockout) to occur. He has estimated that the annual cost of backordering will be 25% of the value of inventory.

(i) What should be the optimum number of units of the product he should buy in one lot ?

(ii) What quantity of the product should be allowed to be back-ordered, if any ?

(iii) What would be the maximum quantity of inventory at any time of the year ?

(iv) Would you recommend to allow back-ordering ? If so, what would be the annual cost saving by adopting the policy of back-ordering. **[JNTU (Mech. & Prod.) 2004; Delhi (M. Com.) 94]**

Solution. In the usual notations, we are given :

D = 10,000 units, C_0 = Rs. 10 per order, C_h = 20% of Rs. 20 = Rs. 4 per unit per year and C_b = 25% of Rs. 20 = Rs. 5 per unit per year.

(i) Economic order quantity (Q *), (a) when stockouts are not permitted :

$$Q^* = \sqrt{\frac{2DC_0}{C_h}} = \sqrt{\frac{2 \times 10{,}000 \times 10}{4}} = 223{\cdot}6 \text{ units}$$

(b) When back-ordering is permitted :

$$Q^* = \sqrt{\frac{2DC_0}{C_h}\left(\frac{C_h + C_b}{C_b}\right)} = \sqrt{\frac{2 \times 10{,}000 \times 10}{4}\left(\frac{4+5}{4}\right)} = 300 \text{ units}$$

(ii) Optimum quantity of the product to be back-ordered is given by :

$$S^* = Q^*\left(\frac{C_h}{C_h + C_b}\right) = 300 \times \left(\frac{4}{4+5}\right) = 133 \text{ units.}$$

(iii) Maximum inventory level, $M^* = Q^* - S^* = 300 - 133 = 167$ units

(iv) Minimum total variable inventory cost in the cases (a) and (b) are :

$$TC\,(223{\cdot}6) = \sqrt{2DC_0C_h} = \sqrt{2 \times 10{,}000 \times 10 \times 4} = \text{Rs. } 894{\cdot}43$$

$$TC\,(300) = \sqrt{2DC_0C_h\left(\frac{C_b}{C_h + C_b}\right)}$$

$$= \sqrt{2 \times 10{,}000 \times 10 \times 4\left(\frac{5}{4+5}\right)} = \text{Rs. } 666{\cdot}67$$

Since TC (223·6) > TC (666·67), the dealer should accept the proposal for back-ordering as this will result in a saving of (894·43 – 666·67) = Rs. 227·76 per year.

Example 13. *Purchase Manager has decided to place order to a minimum quantity of 500 numbers of a particular item in order to get a discount of 10%. From the past records, it was found out that in the last year, 8 orders each of size 200 units were placed. Given the ordering cost* = Rs. 500 *per order, inventory carrying cost of 40% of the inventory value and the price of the item of Rs. 400 per unit. Is the Purchase Manager justified in his decision ? What is the effect of his decision to the company ?*

[JNTU (Mech. & Prod.) 2004; Delhi (FMCI) 2000, (M. Com.) 99]

Solution. In the usual notations, we are given :

D = 8 × 200 or 1,600 units per year, C_0 = Rs. 500 per order.

C = Rs. 400 per unit and C_h = 40% of Rs. 400 = Rs. 160

$$\text{(i) } EOQ, Q^* = \sqrt{\frac{2 \times 1600 \times 500}{160}} = 100$$

Total inventory cost as per EOQ Approach :

$$TC\,(100) = 1600 \times 400 + \frac{1600}{100} \times 500 + \frac{100}{2} \times 160 = \text{Rs. } 656{,}000 \qquad ...(1)$$

(ii) *Total cost as per present policy, i.e.*, for order size of 200 numbers

$$TC\,(200) = 1600 \times 400 + \frac{1600}{200} \times 500 + \frac{200}{2} \times 160 = \text{Rs. } 660{,}000 \qquad ...(2)$$

(iii) Proposed inventory costs : If the manager decides to place an order for a minimum quantity of 500 items to avail a discount of 10%, the cost per items becomes Rs. 360.

$$TC\,(500) = 1600 \times 360 + \frac{1600}{500} \times 500 + \frac{500}{2} \times (160 \times 0{\cdot}4) = \text{Rs. } 613{,}600 \qquad ...(3)$$

From (1), (2) and (3) above, it is evident that if the manager decides to place an order for a minimum quantity of 500 items, he will save Rs. (6,60,000 – 6,13,600) = Rs. 46,400 over the present policy of placing an order of size 200 numbers. Also he will save Rs. (6,56,000 – 6,13,600) = 42,400 over the policy of placing order of economic size, *i.e.*, an order of size 100 numbers.

Hence the purchase manager's decision is justified and it will save Rs. 46,400.

Example 14. *(a) The annual demand for a product is 64,000 units (or 1280 units per week). The buying cost per order is Rs. 10 and the estimated cost of carrying one unit in stock for a year is 20%. The normal price of the product is Rs. 10 per unit. However, the supplier offers a quantity discount of 2% on an order of at least 1000 units of a time, and a discount of 5% if the order is for at least 5,000 units.*

Suggest the most economic purchase quantity per order. **[Delhi (M. Com.) 95]**

Solution. (a) In the usual notations, we are given :

D = 64,000 units, C_0 = Rs. 10 per order, I = 20%, P_1 = Rs. 10,

P_2 = Rs. 9·80 (2% discount on Rs. 10), P_3 = Rs. 9·50.

$b_1 = 1000,\ b_2 = 5000.$

The highest discount available is Rs. 9·50. Thus calculating Q_3^* corresponding to this range is as follows :

$$Q_3^* = \sqrt{\frac{2DC_0}{IP_3}} = \sqrt{\frac{2 \times 64{,}000 \times 10}{0{\cdot}20 \times 9{\cdot}50}} \simeq 821 \text{ units.}$$

Since $Q_3^* < b_2$ (= 5000), it is not feasible.

$$Q_2^* = \sqrt{\frac{2DC_0}{IP_2}} = \sqrt{\frac{2 \times 64{,}000 \times 10}{0{\cdot}20 \times 9{\cdot}80}} = 808 \text{ units}$$

Since $Q_2^* < b_2$ and b_1 calculate Q_1^* and compare total inventory cost corresponding to Q_1^*, b_1 and b_2.

$$Q_1^* = \sqrt{\frac{2DC_0}{IP_1}} = \sqrt{\frac{2 \times 64{,}000 \times 10}{0{\cdot}20 \times 10}} = 800 \text{ units}$$

$$TC(Q_1^*) = DP_1 + C_0 \times \frac{D}{Q_1^*} + i \times P_1 \times \frac{Q_1^*}{2}$$

$$= 64{,}000 \times 10 + 10 \times \frac{64{,}000}{800} + 10 \times 0{\cdot}20 \times \frac{800}{2} = \text{Rs. } 641{,}600.$$

$$TC(b_1) = 64{,}000 \times 9{\cdot}80 + 10 \times \frac{64{,}000}{1{,}000} + 9{\cdot}80 \times 0{\cdot}20 \times \frac{1{,}000}{2} = \text{Rs. } 628{,}820.$$

$$TC(b_2) = 64{,}000 \times 9{\cdot}50 + 10 \times \frac{64{,}000}{5{,}000} + 9{\cdot}50 \times 0{\cdot}20 \times \frac{5{,}000}{2} = \text{Rs. } 612{,}878.$$

Since $TC(b_2) < TC(b_1) < TC(Q_1^*)$, the optimum order size is 5,000 units.

(b) A company uses 8,000 units of a product per year, costing Rs. 10 per unit. The administrative costs per purchase are Rs. 40. The holding costs are 28% of the unit price of the product. The company is following E.O.Q. purchase policy. The company is offered a discount of 1% if the total requirement is purchased four times in a year only, should the offer be accepted. **[JNTU (B. Tech.) 2003]**

Solution. Proceed as in part (a).

Example 15. *A shopkeeper has a uniform demand of an item @ 50 items per month. He buys from a supplier at a cost of Rs. 6/– per item and the cost of ordering is Rs. 10/– each time. If the stock holding costs are 20% per year of stock value, how frequently should he replenish his stock ? Suppose the supplier offers a 5% discount on orders between 200 and 999 items and a 10% discount on orders exceeding or equal to 1000, can the shopkeeper reduce his cost by taking advantage of either of these discounts ?* **[Nagpur (M.B.A.) Nov. 96]**

Solution. In the usual notations, we are given :

D = 50 items per months or 600 items per year.

C_0 = Rs. 10 per order, I = 0·20 and C = Rs. 6 per item.

$$EOQ, (Q^*) = \sqrt{\frac{2DC_0}{C_h}} = \sqrt{\frac{2 \times 600 \times 10}{0{\cdot}20 \times 6}} = 100 \text{ units.}$$

With discount price, we write

Range quantity	Unit purchasing cost
$0 \le Q < 200$	Rs. 6 per unit
$200 \le Q < 1000$	Rs. 6 × 0·95 = Rs. 5·7 per unit
$1{,}000 \le Q$	Rs. 6 × 0·90 = Rs. 5·4 per unit

where Q is the order quantity (lot size).

Using the lowest unit price of Rs. 5·4, we get

$$EOQ, Q_1^* = \sqrt{\frac{2 \times 600 \times 10}{5 \cdot 4 \times 0 \cdot 20}} = 105 \cdot 4 \text{ items.}$$

But it it not feasible because the unit price of Rs. 5·4 is not available for an order size of 105 items. Now, with the unit price equal to Rs. 5·7, we get

$$EOQ, Q_2^* = \sqrt{\frac{2 \times 600 \times 10}{5 \cdot 7 \times 0 \cdot 20}} = 102 \text{ units. This again is not feasible.}$$

With the unit price of Rs. 6, we have

$$EOQ, (Q_2^*) = \sqrt{\frac{2 \times 600 \times 10}{6 \times 0 \cdot 20}} = 100 \text{ units.}$$

This is a feasible order quantity.

Now, the total cost corresponding to 100 units.

$$TC(100) = 6 \times 600 + 10 \times \frac{600}{100} + 6 \times 0 \cdot 2 \times \frac{100}{2} = \text{Rs. } 3720.$$

We shall determine the total cost at cut off points 200 and 1000 also as given below :

$$TC(200) = 5 \cdot 7 \times 600 + 10 \times \frac{600}{200} + 5 \cdot 7 \times 0 \cdot 2 \times \frac{200}{2} = 3534.$$

$$TC(1{,}000) = 5 \cdot 4 \times 600 + 10 \times \frac{600}{1{,}000} + 5 \cdot 4 \times 0 \cdot 2 \times \frac{1{,}000}{2} = 3786.$$

Since $TC(200) < TC(Q_1)$, and also $TC(200) < TC(1{,}000)$, the optimum inventory cost is associated to $Q = 200$ units of ordering quantity. Hence the discounting facility of 5% only is availed by the shopkeeper for selecting the order size as 200 units.

Example 16. *Consider an item for which the following data are available :*

Annual average demand for an item	= 10,000 units
Standard deviation of demand per week	= 25 units
Unit cost	= Rs. 20
Average ordering cost	= Rs. 100 per order
Inventory carrying cost	= 30 per cent
Average lead time	= 4 weeks
Maximum delay	= 2 weeks
Probability of delay	= 0·25
Service level	= 95 per cent

Design an appropriate inventory system for this item. **[Delhi (MBA) March 99]**

Solution. *Fixed order system* :

$$E.O.Q.\ (Q^*) = \sqrt{\frac{2DC_0}{C_h}} = \sqrt{\frac{2 \times 10{,}000 \times 100}{25 \times 0 \cdot 30}} \simeq 516 \text{ units.}$$

Buffer stock= Average demand during lead time

$$= \left(\frac{10{,}000}{52}\right) \times 4 \simeq 769 \text{ units.}$$

Safety stock= Average demand during maximum delay × Probability of delay

$$= \left(\frac{10{,}000}{52}\right) \times 2 \times 0 \cdot 25 \simeq 96 \text{ units.}$$

Reserve stock = Standard deviation of demand during average lead time × k

$$= 25\sqrt{4} \times 1 \cdot 64 = 82 \text{ units.}$$

Reorder point = Buffer + Safety + Reserve = 947 units.

Periodic revised system :

$$\text{Review period} = \frac{Q^*}{D} = \left(\frac{516}{10{,}000}\right) \times 52 = 2 \cdot 6832 \simeq 3 \text{ weeks.}$$

$$\text{Buffer stock} = \left(\frac{10{,}000}{52}\right) \times 3 = 578.$$

$$\text{Safety stock} = \left(\frac{10{,}000}{52}\right) \times 2 \times 0{\cdot}25 = 96 \text{ units.}$$

$$\text{Reserve stock} = 25\sqrt{3} \times 1{\cdot}64 = 71 \text{ units.}$$

$$\text{Re-order point} = \text{Buffer} + \text{Safety} + \text{Reserve} = 745 \text{ units.}$$

EXAMINATION PROBLEMS (ON EOQ MODEL I)

1. A manufacturer has to supply his customer with 600 units of his product per year. Shortages are not allowed and storge cost amounts to 60 paise per unit per year. The set up cost per run is Rs. 80.00. Find :
   (i) the economic order quantity, (ii) the minimum average yearly cost, *(iii)* the optimum number of orders per year
   (iv) the optimum period of supply per optimum order, *(v)* the increase in the total cost associated with ordering (a) 20% more and (b) 40% less than EOQ.

   [**Hint.** R = 600 units/yr., C_1 = Re. 0.60 per unit per year, C_3 = Rs. 80 per production run (i) Use formula (2.15), (ii) Use formula (2.16), (iii) N^* = demand/EOQ , (iv) $t^* = 1/N^*$ (v) *Ordering 20% higher than* EOQ :
   Ordering quatity = (120/100) × 400 = 480 units. With $q^* = 400$, $q = 480$, $p = 480/400 = 1.2$, we have
   $$\frac{C(q)}{C(q^*)} = \frac{1}{2}\left(p + \frac{1}{p}\right) = \frac{1}{2}\left(\frac{1}{1.2} + 1.2\right) = \frac{61}{60}$$
   Thus the cost would increase by 1/60th or 240 × (1/60) = Rs. 4.]

   [**Ans.** (i) q^* = 400 units, (ii) $C_{\min}$ = Rs. 240, (iii) 3/2, (iv) 2/3 yr. = 8 months (v) Rs. 4.]

2. The annual demand for an item is 3200 units. The unit cost is Rs. 6/– and inventory carrying charges 25% per annum. If the cost of one procurement is Rs. 150/–, determine :
   (i) Economic order quantity, (ii) number of orders per year, (iii) time between two consecutive orders, (iv) the optimal cost.
   [JNTU (B. Tech.) 2003]

   [**Hint.** $R = 3200$, $I = 0.25$, C = Rs. 6, C_3 = Rs. 150. Use formulae : (2.15), $t^* = q^*/R$, $N^* = 1/t^*$.]

   [**Ans.** (i) 800 units, (ii) 4, (iii) 3 months, (iv) $C_{\min} = 6R + \sqrt{2RC_1C_3}$ = Rs. 20,400.]

3. A purchase manager places order each time for a lot of 500 numbers of a particular item. From the available data the following results are obtained :
   Inventory carrying cost = 40%, ordering cost per order = Rs. 600 Cost per unit = Rs. 50, Annual demand = 1000.
   Find out the loss to the organization due to his ordering policy.
   [**Ans.** Rs. 1262]

4. Calculate EOQ from the following data :
   Ordering cost Rs. 7; Inventory carrying cost 15%; Unit price Rs. 5; Annual demand 7,500 units. What would be EOQ if the annual sales are Rs. 3000 and Rs. 6,000 ?

5. A company uses annually 12,000 units of a raw material costing Rs. 1.25 per unit. Placing each order costs 45 paise and the carrying costs are 15% *per year per unit* of the average inventory. Find the Economic Order Quantity.
   [JNTU (B. Tech.) 2003]

   [**Hint.** R = 12000, $C_1 = IP$ = Re. 0.15 × 1.25, C_3 = Re. 0.45. Use formula (2.15) to find q^*.

   [**Ans.** The required number of units q^* of raw material = 243 units.]

6. The annual requirements for a particular raw material are 2,000 units, costing Re. 1 each to the manufacturer. The ordering cost is Rs. 10.00 per order and the carrying cost 16% *per annum* of the average inventory value. Find the the Economic Order Quantity and the total inventory cost per annum.
   [**Hint.** R = 2000, C_3 = Rs. 10, $C_1 = IP$ = 0.16 × 1 per unit of quantity per unit time. Use formula (2.15) and (2.16)].

   [**Ans.** q^* = 500 units, $C_{\min}$ = Rs. 80]

7. A certain item costs Rs. 235 per ton. The monthly requirements are 5 tons, and each time the stock is replenished, there is a set-up cost of Rs. 1000. The costs of carrying inventory has been estimated at 10% of the *value of the stock per year.* What is the optimum order quantity.
   [**Hint.** R = 5 tons/month = 60 tons/year, C_3 = Rs. 1000,
   C_1 = 10% of the value of the stock per year = Rs. 235 × (10/100) = Rs. 23.5 per item per year.]

   [**Ans.** q^* = 71.5 tons]

8. An oil engine manufacturer purchases lubricants at the rate of Rs. 42 per piece from a vendor, the requirement of these lubricants is 1,800 per year. What should be the order quantity per order, if the cost per placement of an order is Rs. 16 and inventory carrying charges per rupee per year is only 20 paise.
   [Agra 98; Garhwal M.Sc (Stat) 94; Meerut (M.Sc. Maths) 90]

   [**Hint.** R = 1800, C_3 = 16, $C_1 = IP$ = Rs. 42 × 20, optimum inventory quantity of lubricant = q^* = 83 lubricants **Ans.**]

9. For an item, the production is instantaneous. The storage cost of one item is Re 1 per month and the set-up cost is Rs. 25 per run. If the demand is 200 units per month, find the optimum quantity to be produced per set-up and hence determine the total cost of storage and set-up per month.

   [**Hint.** q^* = 100 units, t^* = 15 days, $C_{\min}$ = Rs. 100, Total costs of storage and set-up = 25 + 1 × 100 = Rs. 125 **Ans.**]

10. A contract has a requirement for cement that amounts to 300 bags per day. No shortages are to be allowed. Cement costs Rs 12 per bag, inventory carrying cost is 10% of the average inventory valuation per day and it costs Rs. 20 to purchase order. Find the minimum cost purchase quantity.

[**Hint.** $R = 1800$, $C_1 = 0.10 \times 12$, $C_3 =$ Rs. 20, $q^* = 100$ *bags*, $C_{\min} =$ Rs. 120.00 **Ans.**].

11. A company uses rivets at a rate of 5,000 kg per year, rivets costing Rs. 2.00/kg. It costs Rs. 20 to place an order and carrying cost of inventory is 10% per year. How frequenty should order for rivets be placed and how much.

[**Ans.** $q^* = 2000$ kg. $t^* = 4.8$ months.]

12. A manufacturer requires 15000 units of a part annually for an assembly operation. The manufacturer can produce this part at the rate of 100 units per day, and the setup cost for each production run is Rs. 24. To hold one unit of this part in inventory costs the manufacturer Rs. 5 per year.

Assuming 250 working days per year, what will be the optimum manufacturing quantity ? Prove the formula used.

[**Ans.** $q^* = 600$ units.]

13. (a) A company required 10,000 units of an item per annum. The cost of ordering is Rs. 100 per order. The inventory carrying cost is 20%. The unit price of the item is Rs. 10. Calculate the Economic Order Quantity.

(b) A company requires 1250 units per month of a particular item. Order costs Rs. 50 per order. The carrying cost is 15% per year while unit cost of the item is Rs. 10. Determine the economic lot size and maximum total variable cost.

14. The Star Equipment Company purchases 54,000 bearing assemblies each year at a unit cost of Rs. 40. The holding cost is Rs. 9 per unit per year and the order cost is Rs. 20. What is the economic order quantity ? How many orders will be placed in one year ?

15. A company requires 1250 units per month of a particular item. Order costs Rs. 50 per order. The carrying cost is 15% per year while unit cost of the item is Rs. 10. Determine economic lot size and minimum total variable cost.

16. Kapil Motors purchase 9000 motor spare parts for its annual requirements, ordering one month usage at a time. Each spare part costs Rs. 20. The ordering cost per order is Rs. 15 and the carrying charges are 15% of the average inventory per year.

You have been asked to suggest a more economical purchasing policy for the company. What advice would you offer and how much would it save the company per year ?

[**Ans.** The company should purchase 300 parts at a time instead of ordering 750 parts every month and net saving by adopting optimum policy is = [Rs. 1305 – Rs. 900] or 31%.]

17. A company uses 24,000 units of a raw material which costs Rs. 12.5 per unit. Placing each order costs Rs. 22.5 and the carrying cost is 5.4 per cent per year of the average inventory. Find the economic order quantity, and the total inventory cost (including the cost of material). **[Meerut 99; Marathwada (M.B.A.) 90]**

[**Ans.** $q^* = 4000$ units, cost = Rs. 30,270]

18. A stockist purchases an item at the rate of Rs. 40 per piece from a manufacturer, 2000 units of the item are required per year. What should be the order quantity per order, if the cost per order is Rs. 15 and inventory charges per year are 20 paise ? Also derive the economic order quantity model used above. **[Garhwal M.Sc. (Stat.) 96]**

[**Hint.** Here $P =$ Rs. 40 per piece, $I = 0{\cdot}20$, $C_3 =$ Rs. 15 and $R = 2000$. Use formula for EOQ.]

19. A company, for one of the *A* class items, placed 6 orders each of size 200 in a year. Given ordering cost = Rs. 600, holding cost = 40%, cost per unit = Rs. 40, find out the loss to the company in not operating scientific inventory policy ? What are your commendations for the future ? **[Osmania (M.B.A.) Feb., 98]**

[**Ans.** $Q^* =$ 300 and corresponding min. annual cost = Rs. 4,800
Total cost = Rs. 5,200
Loss to the company = Rs. (5,200 – 4,800) = Rs. 400]

20. A company is presently having a production run of 500 units every 3 months. He is considering a review of its production-lot size decision. The relevant information is given below :
Annual demand of the item : 2,000 units
Rate of production : 8,000 units per year
Set-up cost : Rs. 300 per run
Inventory holding costs : Rs. 1·60 per unit per year

Would you recommend a change in the current production-lot size ? Why ? What will be the cost savings, if any, as a result of the change ? **[Delhi (M. Com.) 97]**
[**Ans.** 3464 units]

21. Amit manufactures 50,000 bottles of tomato ketchup in an year. The factory cost per bottle is Rs. 5, the set-up cost per production run is estimated to be Rs. 90, and the carrying costs on finished goods inventory amount to 20% of the cost per annum. The production rate is 600 bottles per day, and sales amount to 150 bottles per day. What is the optimum production lot size and the number of production runs ?

If the factory costs increase to Rs. 7·50 per bottle, what will be the optimum production lot size ?

[Punjabi (M.B.A.) 96]

22. (a) FANTASY & FUNN Company is producing one of its products as a voodoo doll. It has a fairly constant demand of 40,000 per year. The soft plastic body is the same for all the dolls, but the clothing is changed periodically to suit customer's choice. Production runs for different products require changing the moulds and setting the plastic forming machines, new pattern for the cutters and sewers and some adjustments in the assembly area. The production rate of previous runs has been averaged as 2,000 dolls per day. The set-up costs are estimated at Rs. 350 per production run. A doll that sells for Rs. 2·50 at a retail outlet is valued at Rs. 1·10 when it comes of the production lines. Complete carrying costs for production items are set at 20% of the production cost and are based on the average inventory level. Determine : (i) :Economic production quantity; (ii) number of production runs per year; (iii) maximum inventory level in the warehouse.

Deduce the mathematical model to be used for determining the economic production quantity.

[Nagarjuna (M.B.A.) 96]

(b) Company A wants to know what production cost its major competitor, company B, has assigned to product item p_7. After a bit of investigation, company A has collected the following data about company B's production of item p_7 :

Production lot size : 2600 units, set up cost : Rs. 135.

annual demand : 30,000 units, daily demand : 100 units.

production rate : 200 units per day,

inventory holding costs : 28% of the average value per year.

Company A has further learnt that company B produces according to 'economic lot size' model.

What is the company B's cost of producing product item p_7 ? **[Delhi (M.B.A.) March 99]**

23. A manufacturer has to supply his customer with 24,000 units of his product every year. This demand is fixed and known. Since the unit is used by the customer in an assembly operation and the customer has no storage space for units the manufacturer must supply a day's requirement each day. If the manufacturer fails to supply the required units, the shortage cost is Rs. 2 per unit per month. The inventory carrying cost is Re. 1 per unit per month, and the set-up cost per run is Rs. 3,500. Determine the optimum run size (Q), the optimum level of inventory (S) at the beginning of any period, the optimum scheduling period, and the minimum total expected relevant yearly cost (TC). **[Poona (M.B.A.) May, 98]**
[**Ans.** (i) 4583 units (ii) 433 units (iii) 2·29 months. (iv) 3,000.]

24. A dealer supplies you the following information pertaining to an item of inventory :

Annual demand	:	800 units
Buying cost	:	Rs. 150 per order
Inventory carrying cost	:	Rs. 3 per unit per year
Back-ordering cost	:	Rs. 20 per unit per year

(i) What will be the optimum number of units of the inventory item he should buy in one lot ?
(ii) What quantity he should allow to be back-ordered ?
(iii) What will be the cost savings, if any, resulting from back ordering ?
(iv) What would be the maximum inventory of the item at any time of the year ?
(v) If the dealer wants that no more than 25% of the units can be back-ordered, should the policy of back-ordering be adopted ? **[Delhi (M. Com.), 96]**

26. A Purchase Manager has decided to place an order for a minimum quantity of 500 units of a particular item of inventory in order to get a discount of 10 per cent. Past records reveal that last year, 8 orders, each of 200 units, were placed. The ordering costs amount to Rs. 500 per order, inventory carrying cost is 40% of the inventory value and price of the item is Rs. 400 per unit.

Is the manager justified in his decision ? What will be the effect of his decision to the company ? **[Delhi (M. Com.) 97]**

27. The annual demand for an item of inventory whose price is Rs. 10/– per unit is 2400 units, ordering costs per order is Rs. 350/– and inventory holding costs are 2% per month. The supplier offers a quantity discount of $7\frac{1}{2}$% if the quality ordered is 400 units or more. The rate of discount will be increased to $12\frac{1}{2}$% if the order is for 3,000 units or more. Find out the economic order quantity. **[Delhi (M. Com.) 98]**

28. A company purchases a key raw material of 3,000 kg. a year at Rs. 10 per kg. It wishes to make its purchases on an iptimum basis. The inventory carrying charges of 50 paise per kg, per year and the interest rate of 12 per cent per annum are based on average inventory. The company estimates that it costs Rs. 106 to place an order. What is the economic order quantity and how often the company should order ? How will the policy change, if the supplier offers 10 per cent discount for orders of 1,000 kg. or more ? **[A.I.M.A. (P.G. Dip. in Management) Dec. 98]**

[**Ans.** (i) $q^* = 612$ kg, 5 order/year, Total cost (q^*) = Rs. 31050, (ii) $q = 632$ kg., 3 order per year; total cost (1,000) = Rs. 28,108]

29. Annual demand for an item is 2,400 units. Ordering cost is Rs. 100, inventory carrying charge is 24% of the purchase price per year. Purchase prices are :

p_1 = Rs. 10 for purchasing $Q_1 \leq 500$

p_2 = Rs. 9cdot25 for purchasing $500 \leq Q_2 < 750$

p_3 = Rs. 8·75 for purchasing $750 \leq Q_3$

Determine the optimum purchase quantity. **[Gujarat M.B.A. (Nov.) 96]**

[**Hint.** $Q_3^* = \sqrt{2 \times 100 \times 2{,}400 / 0{\cdot}24 \times 8{\cdot}75} = 478$ units.

Since $478 < 750 = b_3$, we next compute

$$Q_2^* = \sqrt{2 \times 100 \times 2400 / 9{\cdot}75} = 465 \text{ units}$$

Since $465 < 500 = b_2$, we next compute

$$Q_1^* = \sqrt{2 \times 100 \times 2{,}400/0{\cdot}24 \times 9{\cdot}25} = 447 \text{ units.}$$

Now compare the total costs for purchasing $Q_1^* = 447$, $b_2 = 500$ and $b_3 = 750$ units respectively.

$$TC_1 \text{ (for purchasing 447 units)} = 10 \times 2400 + \frac{2400}{447} \times 100 + \frac{447}{2} \times (0{\cdot}24 \times 10) = \text{Rs. } 25{\cdot}085.$$

$$TC_2 \text{ (for } Q_2 = 500) = 9{\cdot}25 \times 2400 + \frac{2400}{500} \times 100 + \frac{500}{2} \times (0{\cdot}24 \times 9{\cdot}25) = \text{Rs. } 23{,}247.$$

$$TC_3 \text{ (for } Q_3 = 750) = 8{\cdot}75 \times 2400 + \frac{2400}{750} \times 100 + \frac{750}{2} \times (0{\cdot}24 \times 8{\cdot}75) = \text{Rs. } 22{,}119{\cdot}50$$

Hence, the economic purchase quantity for this problem is $Q_3^* = 750$ units.]

30. Determine the economic purchase quantity for the following situation :

Annual demand= 10,000 units, ordering cost Rs. 28·80, carrying cost per unit per year= 20% of the unit price. The quantity versus unit price schedule is 0 – 9999 Rs. 2·00, 10,000–19,999 Rs. 1·60 and 20,000 and up Rs. 1·40.

[Bombay (M.M.S.) 94]

[**Hint.** $D = 10{,}000$, $C_0 = 28{\cdot}80$, $C_h = 20\%$ of price.

(i) $0 \le Q < 10{,}000$; Rs. 2·00 (ii) $10{,}000 \le Q < 20{,}000$; Rs. 1·60.

(iii) $20{,}000 \ge \infty$; Rs. 1·40.

$$Q_3^* = \sqrt{\frac{2 \times 28{\cdot}80 \times 10{,}000}{1{\cdot}40 \times 0{\cdot}2}} = 1{,}435$$

Since Q_3^* is in the range of (i), we compute

$$Q_1^* = \sqrt{\frac{2 \times 28{\cdot}80 \times 10{,}000}{2 \times 0{\cdot}2}} = 1{,}200$$

Total cost for b_1, Q_1^* and b_2 are computed as :

$$C(b_1) = C(10{,}000) = 10{,}000 \times 1{\cdot}60 + \frac{1}{2} \times 10{,}000\,(1{\cdot}6 \times 0{\cdot}2) + 28{\cdot}80 \times \frac{10{,}000}{10{,}000} = 17{,}628{\cdot}80.$$

$$C(Q_1^*) = C(1{,}200) = 10{,}000 \times 2 + \frac{1}{2} \times 1{\cdot}200\,(2 \times 0{\cdot}2) + 28{\cdot}80 \times \frac{10{,}000}{1{,}200} = 20{,}360$$

$$C(b_2) = C(20{,}000) = 10{,}000 \times 1{\cdot}4 + \frac{1}{2} \times 20{,}000\,(1{\cdot}40 \times 0{\cdot}2) + 28{\cdot}80 \times \frac{10{,}000}{20{,}000} = 16{,}814{\cdot}40.$$

$C(b_2)$ is the lowest total cost and hence the procurement policy should be 20,000 items at a time.

31. Find the optimum order quantity for a product for which the price breaks are as follows :

Quantity	Unit cost (Rs.)
$0 \le Q_1 < 100$	20 per unit
$100 \le Q_2 < 200$	18 per unit
$200 \le Q_3$	16 per unit

The monthly demand for the proudct is 400 units. The storage cost is 20 per cent of the unit cost of the product and the cost of ordering is Rs. 25·00 per month.

(b) The office manager of Life Insurance Company orders letterhead stationary from an office products company in boxes of 500 sheets. The annual consumption of the insurance company is 6,500 boxes. Annual carrying costs are Rs. 30 per box and ordering costs are Rs. 280 per order. The following discount price schedule is provided by the office products company :

Order quantity (Boxes)	Price (Rs../Box)
200–999	160
1000–2999	140
3000–5999	130
6000 +	120

Determine the optimum order quantity and the corresponding annual inventory cost. **(Delhi (M.B.A.) Nov. 98)**

[**Hint.** $D = 400$ units; $C = $ Rs. 0·20; $C_0 = $ Rs. 25; $Q_3^* = 79$ units $< b_2\ (= 200)$;

$Q_2^* = 75$ units $< b_1\ (= 100)$; $Q_1^* = 70$ units; $TC(b_2) = $ Rs. 6,770;

$TC(b_1) = $ Rs. 7,480; $TC(Q_1^*) = $ Rs. 8,283].

32. (a) A firm uses every year 12,000 units of a raw material costing Rs. 1·25 per unit. Ordering cost is Rs. 15·00 per order and the holding cost is 5% per year of average inventory. (i) Find the economic order quantity. (ii) The firm follows E.O.Q. purchasing policy. It operates for 300 days per year. Procurement time is 14 days and safety stock is 400 units. Find the re-order point, the maximum inventory and average inventory.

(b) A health centre requires 2000 units of a particular drug per month. Each unit of the drug costs Rs. 3 and the average procurement cost per order is estimated to be at Rs. 150. If the inventory carrying cost is 30% of the average inventory valuation per annum, what quantity of the drug should be ordered ? If the procurement lead time is 6 days, what should the reorder level be ? **[Delhi (M.B.A.) Nov. 98]**

[**Ans.** (a) 2,400 units (ii) R.O.P. = $400 + 14\left(\frac{1,200}{300}\right)$ 2,800 units, 100 units]

33. Consider an item for which :
Annual demand = 1,000 units, cost per unit of item = Rs. 5
inventory carrying cost = 30%, maximum delay = 3 weeks,
probability of delay = 0·30
Determine the buffer stock, reserve stock, safety stock and desirable maximum inventory level for this item.
[Delhi (M.B.A.) March 99]

34. A *Q*-system has been designed for an item '*XYZ*' on the basis of the following information :

Annual demand	10,000 units
Unit cost	Rs. 5·00
Inventory carrying cost	20%
Ordering cost	Rs. 200/order
Lead time	1 week
Safety stock	2,000 units
Annual demand	12,000 units
Unit cost	Rs. 4·00
Inventory carrying cost	20%
Ordering cost	Rs. 400/order.
Lead time	0·5 week
Safety stock	1,000 units.

The average safety stock of 2,000 units was maintained throughout the year. After one year, the information and inventory system was examined and it was discovered that the system was developed on wrong data. The correct data were follows : Determine extra cost incurred due to wrong estimation and design of inventory system.

[Delhi (M.B.A.) April 98]

35. Two products are stocked by a company. the company has limited space and cannot store more than 40 units. The demand distributions for the two products are as follows :

For first product		For second product	
Demand	Probability of demand	Demand	Probability of demand
0	0·10	0	0·05
10	0·20	10	0·20
20	0·35	20	0·50
30	0·25	30	0·10
40	0·10	40	0·15

The inventory carrying costs are Rs. 5 and Rs. 10 per unit of the ending inventories for the first and the second product respectively. The storage costs are Rs. 20 and Rs. 50 per unit of the ending storages for the first and second product respectively.

Find the economic order quantities for both the products. **[Delhi (M.B.A.) Dec. 97]**

36. The storage cost of one item is Re. 1 per month and the set-up cost is Rs. 25 per run. If the demand is 200 units per month, find the optimal size of the batch and the best time for the replenishment of inventory. Production is instantaneous. **[Meerut 2002; Delhi (M.B.A.) Dec. 97]**

37. Using the following data, obtain the Economic order Quantity and the total variable cost associated with the policy of ordering quantities of that size.
Animal demand = Rs. 20,000, Ordering cost = Rs. 150 per order,
Inventory carrying cost = 24 % of average inventory value. **[JNTU (B. Tech.) 2003]**

20.13-3. Model I (c) : Economic Lot Size with Finite Rate of Replenishment (EOQ Production Model)

Let C_1 = holding cost per item per unit time, $C_2 = \infty$, *i.e.*shortages are not permitted,
R = number of items required per unit time, *i.e.* production rate is finite, uniform and greater than R.
t = interval between production cycles, $q = Rt$ (the number of items produced per production run)

Find the expressions for (i) *the optimal order quantity* (ii) *reorder point* (iii) *minimum average cost per unit time.*

Method I—Algebraic Method

Referring to *Fig. 20.6*, the inventory of finished goods does not build up immediately to its maximum point Q. Rather it builds up gradually since goods are being produced faster than they are being sold. The mathematical derivation of this model is given below :

The set-up costs referred to previously as ordering costs in the purchase-size ***Model I (a),*** utilizes the same expression $(R/q)\ C_3$, *i.e.* set-up costs = $(R/q)\ C_3$. ...(20.27)

The inventory carrying costs are determined as follows :

Referring to *Figures 20.4* and *20.6*, average inventory and, in turn, inventory carrying costs are different when there is a continuous flow of goods into the inventory than when the entire lot is received at once. In the production run model, inventory is at its maximum size at the time each production run is completed [see *Fig.* 20.6].

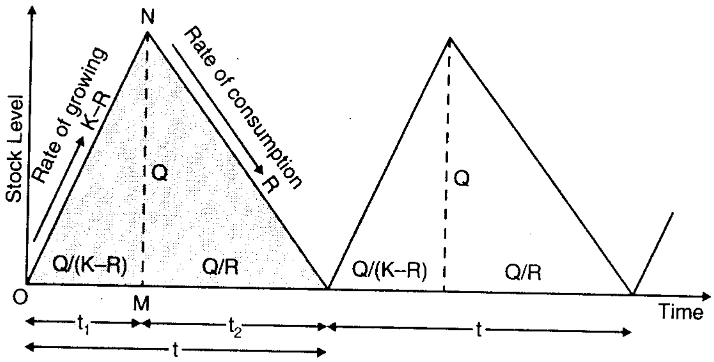


Fig. 20.6 Diagram of inventory level with finite rate of replenishment

The maximum point Q = Time of production run $(t_1) \times$ [production rate (K) – consumption *rate* (R)], *i.e.*

$$Q = (q/K) \times (K - R)\,, \qquad ...(20.28)$$

where the time of production run (t_1) is equal to the optimum number of units produced per run (q) divided by the production rate in units (K), *i.e.* $t_1 = (q/K)$. Then, the average inventory

$$(Q/2) = \tfrac{1}{2}\,(q/K) \times (K - R).$$

Now using the same reasoning as in *EOQ* ***Model I (a),***

$$\begin{pmatrix}\text{average}\\ \text{inventory}\end{pmatrix} \times \begin{pmatrix}\text{carrying}\\ \text{cost}\end{pmatrix} = \begin{pmatrix}\text{total inventory}\\ \text{carrying costs}\end{pmatrix}$$

or

$$\frac{1}{2}\left(\frac{q}{K}\right)(K-R) \times C_1 = \frac{1}{2}\,q\left(\frac{K-R}{K}\right)C_1 \qquad ...(20.29)$$

Now summing up the set-up costs [eqn. (20.27)] and total carrying cost [eqn. (20.29)], we get total inventory cost

$$C(q) = \frac{1}{2}\,q\left(\frac{K-R}{K}\right)C_1 + \left(\frac{R}{q}\right)C_3 \qquad \textbf{(Cost equation)} \qquad ...(20.30)$$

After determining the cost equation [eqn. (20.30)], it should be apparent from our previous derivation of inventory models that manufacturing costs are at their minimum when *the inventory carrying costs* and *the set-up costs become equal.* Thus equating the expressions in eqn. (20.27) and eqn. (20.29), we get

$$\frac{RC_3}{q} = \frac{q}{2}\left(\frac{K-R}{K}\right)C_1 \quad \text{or} \quad q^2 = \frac{2RC_3}{C_1\,(1 - R/K)}$$

or

$$q^* = \sqrt{\frac{2RC_3}{C_1\,(1-R/K)}} \qquad \textbf{(optimal lot size formula)} \qquad ...(20.31)$$

and then $$t^* = \frac{q^*}{R} = \sqrt{\frac{2C_3}{C_1R(1-R/K)}} \quad \textbf{(optimal time interval)} \qquad ...(20.32)$$

Also substituting the value of q from (20.31) in the cost equation [eqn. (20.30)], we get

$$C_{\min} = \sqrt{2C_1\left(1-\frac{R}{K}\right)C_3R} \quad \textbf{(minimum cost formula)} \qquad \textbf{...(20.33)}$$

Cor. 1. The number of cycles per year (N) equals annual usage requirements (R) divided by the economic ordering quantity per production run (q), *i.e.* $N = R/q^*$. Substituting the value of q^* from (20.31), we get

$$N = R\sqrt{\frac{C_1(1-R/K)}{2RC_3}} = \sqrt{\frac{R^2C_1(1-R/K)}{2RC_3}} = \sqrt{\frac{RC_1(1-R/K)}{2C_3}} \qquad ...(20.34)$$

Cor. 2. The equation (20.34) reflects only the quantities and costs on single product produced on the machine. In order to reflect the same for the second product the model must be modified for,

$$N = \sqrt{\frac{R_1C_1^{(1)}(1-R_1/K_1) + R_2C_1^{(2)}(1-R_2/K_2)}{2(C_3^{(1)}+C_3^{(2)})}} \qquad ...(20.35)$$

where the additional subscripts 1 and 2 are used for the first and second products, respectively.

The quantities for the first and second products manufactured per run during the year can be easily determined. The ordering quantities per production run for each product is the annual usage of the particular product divided by the number of production runs annually for each product.

Cor. 3. The formula (20.35) is not restricted to two products only. It can be expanded to include several products. In compact form, the general formula for several products become :

$$N = \sqrt{\frac{\Sigma[R_iC_i^{(1)}(1-R_i/K_i)]}{2\Sigma C_3^{(i)}}} \qquad ...(20.36)$$

where i is used to represent values for each product.

Method II—Calculus Method

In this model, each production cycle time t consists of two parts t_1 and t_2 where —

(i) t_1 *is the period during which the stock is growing up at a rate of* $(K-R)$ *items per unit time.*

(ii) t_2 *is the period during which there is no replenishment* (or *supply or production*), *but there is only a constant demand at the rate of* R.

Further, we assume that Q is the stock available at the end of time t_1 which is expected to be consumed during the remaining period t_2 at the consumption (or demand) rate R.

Now, it is evident from the graphical situation in *Fig. 20.6* that

$$t_1 = Q/(K-R) \text{ and } t_2 = Q/R. \qquad ...(20.37)$$

Adding t_1 and t_2, we get $t_1 + t_2 = Q/(K-R) + Q/R$ or $t = QK/R(K-R)$ ($\because t = t_1 + t_2$)

which gives $$Q = \frac{(K-R)}{K}Rt = \frac{(K-R)}{K}.q \quad (\because q = Rt) \qquad ...(20.38)$$

Now, holding cost for time period $t = (\Delta\, ONB)\, C_1 = \frac{1}{2} MN \times OB \times C_1 = \frac{1}{2} QtC_1$

and the set-up cost for period $t = C_3$.

Therefore, the total average cost is given by $C(q) = \dfrac{\frac{1}{2}QtC_1}{t} + \dfrac{C_3}{t}$ ***(Cost equation)*** ...(20.39)

Now substituting the value of Q from eqn. (20.38) and of t from ($q = Rt$), the cost equation (20.39) becomes

$$C(q) = \frac{1}{2}\left(\frac{K-R}{K}q\right)C_1 + C_3\frac{R}{q}. \qquad ...(20.40)$$

For optimum value of q, we have $\dfrac{dC}{dq} = \dfrac{1}{2}\left(1-\dfrac{R}{K}\right)C_1 - \dfrac{C_3R}{q^2} = 0$

which gives $$q = \sqrt{\left(\frac{2C_3}{C_1}.\frac{RK}{K-R}\right)}.$$

Since $\frac{d^2C}{dq^2} = 0 + \frac{2C_3R}{q^3} > 0$, we have

$$q^* = \sqrt{\left(\frac{2C_3}{C_1}.\frac{RK}{K-R}\right)} \quad \textbf{(Optimal lot size formula)} \quad ...(20.41)$$

and

$$t^* = \frac{q^*}{R} = \sqrt{\frac{2C_3K}{C_1R\,(K-R)}} \quad \textbf{(Optimal time interval)} \quad ...(20.42)$$

Also, substituting the value of q^* from (20.41) in equation (20.40) and simplifying, we get

$$C_{\min} = \sqrt{2C_1\left(1-\frac{R}{K}\right)C_3R} \quad ...(20.43)$$

We now observe that—

(i) if $K = R$, than $C_{\min} = 0$, which implies that there will be no carrying cost and no set-up cost;

(ii) if $K \to \infty$, *i.e.* the production rate is infinite, then this problem becomes exactly the same as ***Model I*** **(a)**;

(iii) although, in this model, C_3 is same as in ***Model I (a)*** and ***I (b)***, but the carrying cost is reduced in the ratio $(1 - R/K) : 1$ for minimum cost.

Q. 1. Derive the optimal economic lot size formula $q = \sqrt{\left(\frac{2C_3RK}{C_1\,(K-R)}\right)}$ in the usual notations when the rate of replenishment is finite. Also, derive the minimum cost formula $C_{\min} = \left[2C_1\left(1-\frac{R}{K}\right)C_3R\right]^{1/2}$.

2. What are the assumptions of the basic inventory model ? How does each affect the model ? **[Madras (M.B.A.) 90]**

3. Obtain the economic order quantity with finite rate of replenishment and the minimum cost. **[Garhwal M.Sc. (Stat.) 93]**

4. Derive an expression for economic production quantity with uniform rate of repleshment with no shortages. **[JNTU (B. Tech.) 2003]**

20.13–4. Limitations on EOQ Formulae

In spite of several assumptions made in the derivation of above EOQ formulae, following limitations should be considered for applications of these formulae :

1. The demand is neither known with certainty nor it is uniform in practical situations. If there are slight fluctuations, the formula is practically valid. But, if the fluctuations are considerable, the formula loses its validity.
2. It is difficult to measure the ordering cost and, also, it is not linearly related to the number of orders and rises in stepped manner with the increasing number of orders. The inventory carrying rate I is even more difficult to measure. So it is simply assumed at 20%, though if patiently estimated for a given situation, it may lie between 7% to 35%.
3. In EOQ models, it is assumed that the annual demand can be estimated in advance. But, annual demand cannot be simply estimated with accuracy. For example, demand per week may be so erratic that one would hesitate to predict annual demand from it. In another situation, one may be dealing with a brand new product which has absolutely no previous history. So, to estimate annual demand is not always possible. Thus EOQ formulae cannot be used under such circumstances.
4. In EOQ model, it is assumed that the inventory rises to its maximum level instantaneously. But, in many cases it may not be true because the orders may be delivered in portions over a period of time. In such cases, inventory is being used while new inventory is still being received, and the inventory does not build up immediately to maximum level. So EOQ model is not valid under such conditions.
5. It is also assumed in EOQ models that the proper use of estimating, accounting and other cost finding techniques can determine the appropriate values of carrying cost and ordering cost. Sometimes,

proper records are not maintained by the organisation which could provide sufficient data base to generate the cost parameters.

6. In EOQ, it is assumed that the demand is uniform. But, uniformity is seldom observed in practical situations. In several cases, it may be less or more. All the items of inventory may be withdrawn at the beginning of the period, at the end of the period or uniformly throughout the period.
7. In EOQ models, the replenishment time is also assumed to be zero. But, this is not possible until unless the supplier is nearby. In the case of the made-ins, this assumption does not hold good. Replenishment time is usually not euqal to zero, and it may constitute a significant fraction of time.
8. If the EOQ models are applied without due regard to the possibility of a falling demand, then it can lead to a high value of obsolenscence inventory.

Q. 1. Derive the Wilson's EOQ formula. What are the practical limitations on the EOQ formula ? Discuss its sensitivity.

2. "Although the classic inventory decisions model, known as the EOQ model, is too over simplified to represent many of the real world situations, it is the excellent starting point from which to develop more realistic and complex inventory decision models". In the light of above statement explain the major limitations of the EOQ model. Also explain the major techniques developed by management scientists to overcome these limitations.

3. With the help of a Quantity-Cost curve, explain the significance of Economic Order Quantity. What are the limitations in using the formula for EOQ ?

Example 17. *A contractor has to supply 10,000 bearings (see Fig. 20.7) per day to an automobile manufacturer. He finds that, when he starts a production run, he can produce 25,000 bearings per day. The cost of holding a bearing in stock for one year is 20 paise, and set-up cost of a production run is Rs. 180.00. How frequently should production run be mode.* **[Agra 98; Meerut (Maths) 93 P; Rohilkhand 90]**

Solution. Proceeding as in ***Mode I*** **(c)**, we obtain the expressions

$$q^* = \sqrt{\left(\frac{2C_3RK}{C_1\,(K-R)}\right)} = \sqrt{\frac{2C_3}{C_1}} \cdot \sqrt{\frac{RK}{K-R}} \qquad \text{...(20.44)}$$

and

$$t^* = \sqrt{\left(\frac{2C_3K}{RC_1\,(K-R)}\right)} = \sqrt{\frac{2C_3}{RC_1}} \cdot \sqrt{\frac{K}{K-R}} \qquad \text{...(20.45)}$$

In usual notations, we are given that

C_1 = Re. 0.20 per bearing per year = Re. $\left(\frac{0.20}{365}\right)$ per bearing per day = Re. 0.00055

C_2 = Rs. 180.00 per production run, R = 10,000 bearings per year K = 25,000 bearings per day.

Substituting these values in expressions (20.44) and (20.45), we get

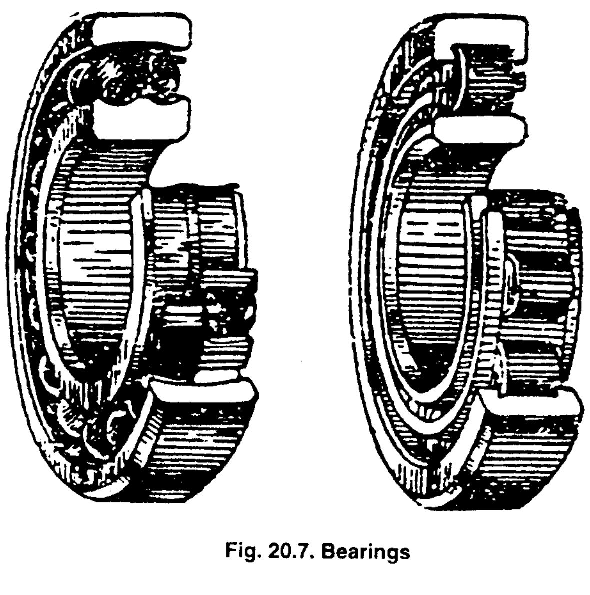


Fig. 20.7. Bearings

$$q^* = \sqrt{\frac{2 \times 180 \times 10{,}000 \times 25{,}000}{0.00055 \times (25{,}000 - 10{,}000)}} = \sqrt{1.09 \times 10^{10}} = 1{,}05{,}000 \text{ bearings}$$

and $$t^* = \sqrt{\left(\frac{2C_3K}{RC_1(K-R)}\right)} = \sqrt{\frac{2 \times 180 \times 25000}{10000 \times 0.00055 \times 15000}} = 0.3 \text{ day}.$$

Example 18. *Amit manufactures 50,000 bottles of tomato ketch-up in an year. The factory cost per bottle is Rs. 5, the setup cost per production run is estimated to be Rs. 90, and the carrying costs on finished goods inventory amount to 20% of the cost per annum. The production rate is 600 bottles per day, and sales amount to 150 bottles per day. What is the optimal production lot size and the number of production runs ?* **[Delhi (M.Com.) 90]**

Solution. Here C = Rs. 5, I = 20%, K = 600 units per day, R = 150 bottles per day, C_3 = Rs. 90.

Proceed as above *Example* 17.

Example 19. *In a paints manufacturing unit, each type of paint is to be ground to a specified degree of fineness. The manufacturer uses the same ball mill for a variety of paints and after completion of each batch, the mill has to be cleaned and the ball charge properly made up. The change-over from one type of paint to another, is esimated to cost Rs. 80 per batch. The annual sales of a particular grade of paint are 30,000 litres and the inventory carrying cost is Re. 1 per litre. Given that the rate of production is 3 times the sales rate, determine the economic batch size.*

Solution. Here R = 30,000, C_3 = Rs. 80, C_1 = Re. 1, K = 90,000,

$$q^* = \sqrt{\frac{2 \times 30{,}000 \times 80}{1.00\,[1 - (30{,}000/90{,}000)]}} = 2683.28 \textbf{ Ans.}$$

Number of batches per year = 30,000/2683.28 = 11.18 **.Ans.**

EXAMINATION PROBLEMS (ON EOQ MODEL I (C))

1. Find the most economic batch quantity of a product on a machine if the production rate of that item on the machine is 200 pieces/day and the demand is uniform at the rate of 100 pieces/day. The setup cost is Rs. 200/– per batch and the cost of holding one item in inventory is Re 0.81 per day. How will the batch quantity vary if the machine production rate was infinite ?
2. Assuming you are reviewing the production lot size decision associated with a production operation where the production rate is 8000 units a year, annual demand is 2000 units, setup cost is Rs. 300/– per production run and holding cost is Rs. 1.60 per unit per year. The current production run is 500 units every 3 months.

   Would you recommend a change in the production lot size ? If so, why ? How much could be saved by adopting the new production run lot size ?
3. You have been given the following information regarding the production lot size of a particular product :

   Annual demand = 5,000 units, Setup cost = Rs. 100 per set-up, Daily demand = 17 units, Production rate = 50 units per day, Optimum production lot size = 275 units.

   Rising interest rates have caused a 10% increase in the holding costs. Determine the new optimum production lot size for the product.
4. Fantasy & Funn Company is producing one of its products as a voodoo doll. It has a fairly constant demand of 40,000 per year. The soft plastic body is the same for all the dolls, but the clothing is changed periodically to suit customers' choice. Production runs for different products require changing the moulds and setting the plastic forming machines, new patterns for the cutters and sewers and some adjustments in the assembly area. The production rate of previous runs has been averaged as 2,000 dolls per day. The setup costs are estimated at Rs. 250/– per production run. A doll that sells for Rs. 2.50 at a retail outlet is valued at Rs. 1.10 when it comes off the production line. Complete carrying costs for production items are set at 20% of the production cost and are based on the average inventory level. Determine :

   (i) Economic production quantity; (ii) Number of production runs per year, (iii) Maximum inventory level in the warehouse. Also deduce the mathematical model to be used for determining the economic production quantity.
5. An item is produced at the rate of 50 items per day. The demand occurs at the rate of 25 items per day. If the setup cost is Rs. 100 per run and inventory cost is Rs. 0·01 per unit of item per day. Find the economic lot size for one run, assuming that shortages are not permitted. **[Virbhadrah 2000; Garhwal M.Sc. (Stat.) 95, 92]**

20.14. THE EOQ MODEL WHEN SHORTAGES ARE ALLOWED

20.14–1. Model II (a) : The EOQ with Constant Rate of Demand, Scheduling Time Constant

Model II *is the extension of* ***Model I*** *allowing shortages. We discuss this model in three parts, the assumptions in each part being slightly changed.*

Implied in this model are the following assumptions :

(i) ***C_1 is the holding cost per quantity unit per unit time.***

(ii) ***C_2 is the shortage cost per quantity unit per unit time.***

(iii) R quantity per unit time is the demand rate.

(iv) t_p is the scheduling time period which is constant.

(v) q_p is the fixed lot size ($q_p = Rt_p$).

(vi) z is the order level to which the inventory is raised (planned) in the beginning of each scheduling period. Shortages, if any, have to be made-up. Here z is the variable.

(vii) Production rate is infinite.

(viii) Lead time is zero.

Determine : (a) Optimal order level z. (b) Minimum average cost.

Solution. In this model, we can easily observe that the inventory carrying cost C_1 as well as shortage cost C_2 will be involved only when $0 \le z \le q_p$.

In the *Fig. 20.8*, the dotted area (ΔDBC) represents the failure to meet the demand and the shady area (ΔAOB) shows the inventory.

Since q_p is the lot size sufficient to meet the demand for time t_p, but ($< q_p$) amount of stock is planned in order to meet the demand for time z/R (R is the demand rate), shortage of amount $(q_p - z)$ will arise for the entire remaining period $(t_p - z/R)$.

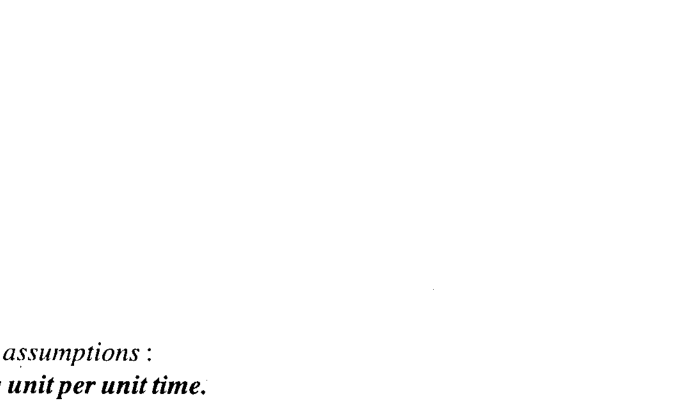


Fig. 20.8 Graphical representation of EOQ model with shortages.

Thus, holding cost per unit time becomes

$$= C_1\,(\Delta OAB)/t_p = \frac{C_1}{t_p}\left(\frac{1}{2}\cdot z\cdot\frac{z}{R}\right) = \frac{1}{2}\frac{z^2C_1}{q_p} \quad (\text{since } Rt_p = q_p) \qquad \text{...(20.46)}$$

and shortage cost per unit time is

$$= C_2\,(\Delta BDC)/t_p = C_2\,(½.\,BD\,.\,DC)/t_p = \frac{C_2}{2t_p}\left(t_p - \frac{z}{R}\right)(q_p - z) = \frac{C_2}{2Rt_p}(Rt_p - z)(q_p - z) = \frac{1}{2}\frac{C_2}{q_p}(q_p - z)^2. \qquad \text{...(20.47)}$$

Now, adding (20.46) and (20.47), we get total *average cost*

$$C(z) = \frac{1}{2}\frac{z^2C_1}{q_p} + \frac{1}{2}\frac{C_2}{q_p}(q_p - z)^2 \quad \textbf{(Cost equation)} \qquad \text{...(20.48)}$$

Note. Since set-up cost C_3 and period t_p are constant, the average set-up cost C_3/t_p also being constant will not be considered in the cost equation.

To obtain the optimum order level z, we differentiate $C(z)$ with respect to z and set the derivative equal to zero. Thus, we get

$$\frac{dC}{dz} = \frac{1}{2}\frac{C_1}{q_p}(2z) + \frac{1}{2}\frac{C_2}{q_p}\cdot 2(q_p - z)(-1) = 0,$$

which gives us,
$$z = \frac{C_2}{C_1 + C_2}\cdot q_p = \frac{C_2}{C_1 + C_2}\cdot Rt_p \quad \textbf{(Optimum order level)} \qquad \text{...(20.49)}$$

Condition for minimum cost is also satisfied, because

$$\frac{d^2C}{dz^2} = \frac{C_1}{q_p} + \frac{C_2}{q_p} > 0.$$

Substituting the value of z from (20.49) in (20.48), we get

$$C_{\min} = \frac{1}{2}\frac{C_1}{q_p}\left(\frac{C_2 q_p}{C_1 + C_2}\right)^2 + \frac{1}{2}\frac{C_2}{q_p}\left(q_p - \frac{C_2 q_p}{C_1 + C_2}\right)^2$$

or
$$C_{\min} = \frac{1}{2}\frac{C_1 C_2}{C_1 + C_2} \cdot q_p = \frac{C_1 C_2}{2(C_1 + C_2)} \cdot Rt_p \quad \textbf{(Minimum cost)} \qquad \text{....(20.50)}$$

Thus unless $C_1 = 0$, optimal order level z in eqn. (2.49) is less than the demand q_p during the scheduling period t_p (because $C_1 + C_2 > C_2$). So we must deliberately plan for shortages because of $z < q_p$.

Q. 1. Consider the following inventory model, in which run size is constant and shortages are allowed with a penalty cost for late deliveries :

C_1 = Cost per day of holding an item in inventory.

C_2 = Penalty cost per day of failing to deliver one item on schedule.

R = Contracted number of item per day.

Determined the inventory level = z at the beginning of a month that minimizes costs.

2. Discuss various inventory costs. Derive economic lot size formula when shortage costs are allowed.
[JNTU (B. Tech.) 2003; Agra 93; Garhwal M.Sc (Math.) 93, 91]

Example 20. *A sub-contractor undertakes to supply diesel engines to a truck manufacturer at the rate of 25 per day. There is a clause in the contract penalizing him Rs. 10.00 per engine per day late for missing the scheduled delivery date. He finds that the cost of holding a completed engine in stock is Rs. 16.00 per month. This production process is such that each month (30 days) he starts a batch of engines through the shops and all these engines are available for delivery any time after the end of the month. What should his inventory level be at the beginning of each month (i.e. immediately after taking in) to stock the engines made in the previous month, and then shipping engines to fill unsatisfied demand from previous month ?* **[Meerut M.Sc. (Maths.) 2000]**

Solution. In this problem, we have $R = 25$ engines per day, $C_1 =$ Rs. $16/30$, $C_2 =$ Rs. 10.00, $t_p = 30$ days.

Now, using the result (20.49), we have

$$z = \frac{C_2}{C_1 + C_2} \cdot Rt_p = \frac{10}{10 + (16/30)} \times 25 \times 30 = 712 \text{ engines.} \qquad \textbf{Ans.}$$

20.14–2. Model II (b). The EOQ with Constant Rate of Demand, Scheduling Time Variable

In a certain manufacturing situation :

(i) R is the demand per year, (ii) the production is instantaneous,

(iii) q (= Rt) is the order quantity per run,

(iv) t is the scheduling time period which is variable [not constant as in ***Model II (a)****],*

(v) z is the order level to which the inventory is raised, and (vi) *lead time is zero.*

Show that the optimal order quantity q per run which minimizes the total cost is

$$q = \sqrt{\frac{2RC_3(C_1 + C_2)}{C_1 C_2}}$$

where $C_1 =$ *storage cost per unit per year,* $C_2 =$ *shortage cost per unit per year,* $C_3 =$ *setup cost per run.*
[Meerut 99; Meerut (Stat.) 98, 90; Garhwal M.Sc (Stat.) 94, 92]

Solution. Since all the assumptions in this model are same as in ***Model II (a)*** except with the difference that the scheduling period t is not constant here, so it now becomes important to consider the average set-up cost C_3/t in the cost equation so that comparison can be made between different values of t.

Thus, in the similar fashion as discussed in ***Model II (a),*** the cost equation becomes

$$C(t, z) = \frac{1}{t}\left(\frac{C_1 z^2}{2R} + \frac{1}{2}\frac{C_2}{R}(Rt - z)^2 + C_3\right) \text{ [see Fig. 20.8 of } \textbf{\textit{Model II}}\textbf{ (a)}] \qquad \text{...(20.51)}$$

Now, we want to minimize the cost $C(t, z)$ which is the function of two independent variables z and t.

For this, we have $\dfrac{\partial C}{\partial z} = 0$, and $\dfrac{\partial C}{\partial t} = 0$.

So differentiating (20.51) partially w.r.t. 'z' and equating to zero, we get

$$\frac{\partial C}{\partial z} = \frac{1}{t}\left(\frac{2C_1 z}{2R} - \frac{2C_2}{2R}(Rt - z)\right) = 0$$

which gives
$$z = \frac{C_2 Rt}{C_1 + C_2}. \qquad \text{...(20.52)}$$

Similarly,
$$\frac{\partial C}{\partial t} = \frac{-1}{t^2}\left(\frac{C_1 z^2}{2R} + \frac{C_2}{2R}(Rt - z)^2 + C_3\right) + \frac{1}{t}\left(0 + \frac{C_2}{2R}\,2\,(Rt - z)\,R + 0\right) = 0$$

which gives
$$-\frac{1}{t^2}\left(\frac{C_1 z^2}{2R} + \frac{C_2}{2R}(Rt - z)^2 + C_3\right) + \frac{C_2}{t}(Rt - z) = 0\,.$$

Multiplying this equation by $2Rt^2$ and simplifying we get
$$-(C_1 + C_2)\,z^2 + C_2 R^2 t^2 = 2RC_3$$

Now to find the value of t, substitute the value of z from (20.52), and get
$$-\frac{R^2 t^2 C_2^2}{C_1 + C_2} + C_2 R^2 t^2 = 2RC_3 \quad \text{or} \quad C_2 R^2 t^2\left(1 - \frac{C_2}{C_1 + C_2}\right) = 2RC_3$$

or
$$C_2 R t^2\left(\frac{C_1}{C_1 + C_2}\right) = 2C_3 \quad \text{or} \quad t = \sqrt{\frac{2C_3\,(C_1 + C_2)}{RC_1 C_2}} \qquad \textbf{(Reorder Time)} \qquad \text{...(20.53)}$$

For minimum cost we may further verify that
$$\frac{\partial^2 C}{\partial t^2}\cdot\frac{\partial^2 C}{\partial z^2} - \left(\frac{\partial^2 C}{\partial t\,\partial z}\right)^2 > 0\,, \text{ and } \frac{\partial^2 C}{\partial t^2} > 0\,,\ \frac{\partial^2 C}{\partial z^2} > 0.$$

Optimal order quantity q is given by
$$q^* = Rt^* = R\sqrt{\frac{2C_3\,(C_1 + C_2)}{RC_1 C_2}} \quad \text{or} \quad q^* = \sqrt{\frac{2RC_3\,(C_1 + C_2)}{C_1 C_2}} \qquad \textbf{(E.O.Q.)} \qquad \text{...(20.54)}$$

which is the required result.

To compute the formula for *minimum cost,* first we substitute the value of z from the result (20.52) in the cost equation (20.51), and get

$$C_{\min} = \frac{C_1}{2Rt}\cdot\frac{C_2^2\,(Rt)^2}{(C_1 + C_2)^2} + \frac{C_2}{2Rt}\left(Rt - \frac{C_2 Rt}{C_1 + C_2}\right)^2 + \frac{C_3}{t}$$
$$= \frac{C_1 C_2^2 Rt}{2\,(C_1 + C_2)^2} + \frac{C_2 Rt}{2\,(C_1 + C_2)^2}(C_1 + C_2 - C_2)^2 + \frac{C_3}{t}$$
$$= \frac{C_1 C_2^2 Rt}{2\,(C_1 + C_2)^2} + \frac{C_2 Rt C_1^2}{2\,(C_1 + C_2)^2} + \frac{C_3}{t}$$
$$= \frac{C_1 C_2 Rt\,(C_1 + C_2)}{2\,(C_1 + C_2)^2} + \frac{C_3}{t} \quad = \frac{C_1 C_2 Rt}{2\,(C_1 + C_2)} + \frac{C_3}{t}\,.$$

Now substituting the value of t from (20.53), we get
$$C_{\min} = \frac{C_1 C_2 R}{2\,(C_1 + C_2)}\cdot\sqrt{\frac{2C_3\,(C_1 + C_2)}{RC_1 C_2}} + C_3\cdot\sqrt{\frac{RC_1 C_2}{2C_3\,(C_1 + C_2)}}$$
$$= \sqrt{\frac{C_1 C_2 C_3 R}{2\,(C_1 + C_2)}} + \sqrt{\frac{C_1 C_2 C_3 R}{2\,(C_1 + C_2)}} \quad = \sqrt{\frac{2C_1 C_2 C_3 R}{(C_1 + C_2)}}$$

or
$$C_{\min} = \sqrt{2C_1 C_3 R}\,\sqrt{C_2/(C_1 + C_2)} \qquad \textbf{(Note)} \qquad \text{...(20.55)}$$

Further, it is interesting to note here that the minimum cost given by (20.55) is less than that already given by (20.16) in ***Model I (a)*** *i.e.,*
$$\sqrt{2C_1 C_3 R}\,\sqrt{C_2/(C_1 + C_2)} < \sqrt{2C_1 C_3 R} \quad \text{or} \quad \sqrt{C_2/(C_1 + C_2)} < 1 \ \text{ i.e. } \ C_2 < C_1 + C_2\,.$$

Hence, in ***Model II (b),*** the cost is further reduced in comparison to ***Model I (a).***

Corollary. If, in addition to the assumptions prescribed for ***Model II (b)***, we are given the production cost b per item, then the cost equation (20.51) becomes

$$C(t, z) = \frac{C_1 z^2}{2Rt} + \frac{1}{2} C_2 \frac{(Rt - z)^2}{Rt} + \frac{C_3 + bRt}{t} \qquad \text{...(20.56a)}$$

or

$$C(t, z) = \frac{1}{2Rt}[C_1 z^2 + C_2 (Rt - z)^2] + \frac{C_3}{t} + bR. \qquad \text{...(20.56b)}$$

For minimizing the cost $C(t, z)$, the derivative of the term bR becomes zero (b and R both being constant), hence the optimum values of t and q will the same as above {(20.53), (20.54)].

Q. 1. Show that for a system where demand is deterministic and is a constant R units per unit time and the production rate is infinite, it is never optimal in comparison to have any lost sales.

[**Hint.** Compare the minimum cost of **Model II (b)**, with the minimum cost of **Model I (a)** as explained earlier.]

2. Discuss the impact on lot size if shortage are allowed.

Example 21. *The demand of an item is uniform at a rate of 25 units per month. The fixed cost is Rs. 15 each time a production run is made. The production cost is Re. 1 per item, and the inventory carrying cost is Re. 0.30 per item per month. If the shortage cost is Rs. 1.50 per item per month, determine how often to make a production run and of what size it should be ?* **[Meerut 99; Agra 98, 93; Banasthali 93]**

Solution. This problem is exactly based on the corollary of ***Model II (b)*** whose cost equation is given by (20.56 a, b). The given values are :

C_1 = Re. 0.30 per item per month $\qquad$ b = Re. 1.00 per item (production cost)

C_2 = Rs. 1.50 per item per month

C_3 = Rs. 15.00 per production run $\qquad$ R = 25 units per month

As explained in above *Corollary*, the optimum values of q and t are obtained by substituting the relevant values in (20.54) and (20.53). Thus, we get

$$q^* = \sqrt{\frac{2RC_3 (C_1 + C_2)}{C_1 C_2}} = \sqrt{\frac{2 \times 15 \times 25\ (0.30 + 1.50)}{0.30 \times 1.50}} = 54 \text{ items.} \qquad \textbf{Ans.}$$

and

$$t^* = \frac{q^*}{R} = \frac{54}{25} \text{ months} \quad = 2.16 \text{ months. } \textbf{Ans.}$$

Example 22. *Given the following data for an item of uniform demand, instantaneous delivery time and back order facility : Annual Demand* = 800 *units, Cost of an Item* = *Rs.* 40, *Ordering Cost* = *Rs.* 800, *Inventory Carrying Cost* = 40% *Back Order Cost* = *Rs.* 10.

Find out : (i) Minimum cost order quantity, (iii) Maximum inventory level,
(ii) Maximum number of back orders, (iv) Time between orders, and (v) Total annual cost.

Solution. We are given the following data for an item with uniform demand, instantaneous delivery time and back order facility : R = 800 units, C = Rs. 40, C_3 = Rs. 800, I = 0.40, C_2 = Rs. 10.

(i) Minimum cost order quantity q is given by

$$q^* = \sqrt{\frac{2RC_3}{CI}\left(\frac{CI + C_2}{C_2}\right)} = \sqrt{\frac{2 \times 800 \times 800}{40 \times 0.40}\left(\frac{40 \times 0.40 + 10}{10}\right)} = 456 \text{ units.} \qquad \textbf{Ans.}$$

(ii) Maximum number of back order quantity is given by

$$z = q^* \left(\frac{CI}{CI + C_2}\right) = 456 \times \frac{16}{16 + 10} = 281 \text{ units.} \qquad \textbf{Ans.}$$

(iii) Maximum inventory level is given by

$$I_{max} = q^* - z = (456 - 281) \text{ units} = 175 \text{ units.}$$

(iv) Time between orders $= \frac{q^*}{R} = \frac{456}{800} = 0.57$ year = 6.84 months. **Ans.**

(v) Total annual cost is given by

$$C_{min} = \frac{RC_3}{q^*} + \frac{(q^* - z)^2}{2q^*} C_2 + \frac{z^2}{2q^*} CI$$

$$= \frac{800}{456} \times 800 + \frac{(456-281)^2}{2 \times 456} \times 10 + \frac{281^2}{2 \times 456} \times 16 \text{Rs. } 2807.$$ **Ans.**

Example 23. *The demand for an item is deterministic and constant over time and it is equal to 600 units per year. The per unit cost of the item is Rs. 50 while the cost of placing an order is Rs. 5. The inventory carrying cost is 20% of the cost of inventory per annum and the cost of shortage is Re. 1 per month. Find the optimal ordering quantity when stockouts are permitted. If the stockouts are not permitted, what would be the loss to the company.*

Solution. We are given that $R = 600$ units, $C_1 = 0.20 \times 50 =$ Rs. 10, $C_3 =$ Rs. 5 per order, and $C_2 =$ Re. 1 per unit/month or Rs. 12 per year/unit.

(i) When stockouts are permitted, the optimal ordering quantity is given by

$$q^* = \sqrt{\frac{2RC_3}{C_1}\left(\frac{C_1 + C_2}{C_2}\right)} = 33 \text{ units.}$$

Maximum number of back-orders, $z^* = q^*\left(\frac{C_1}{C_1 + C_2}\right) = 33\left(\frac{10}{10+12}\right) = 15$ units.

Total expected yearly cost (with shortages allowed) is given by :

$$C(q^*) = \sqrt{2RC_1C_3\left(\frac{C_2}{C_1 + C_2}\right)} = 180.91 \approx 181.$$

(ii) If stockouts or back-orderings are not permitted, the optimal order quantity is

$$q_1^* = \sqrt{\frac{2RC_3}{C_1}} = 24.5 \text{ units.}$$

The total relevant cost is given by $C(q_1^*) = \sqrt{2C_1C_3R} =$ Rs. 245.

Thus, the additional cost when back-ordering is not allowed is = Rs. (245 − 181) = Rs. 64.**Ans.**

EXAMINATION PROBLEMS

1. Consider the following data :

   Unit Cost = Rs. 100, Order Cost = Rs. 160, Inventory Carrying Cost = Rs. 20. Back-order Cost (stock-out cost) = Rs. 10, Annual Demand = 1000 units. Compute the following :

   (i) Minimum cost order quantity, (ii) Time between orders, (iii) Minimum number of back-orders, (iv) maximum inventory level and (v) Overall annual cost. **[C.A. (May) 90]**

   [**Ans.** (i) 219, (ii) 219, (iii) 146, (iv) 73, (v) 1460.]

20.14–3. Model II (c) : The production Lot Size Model with Shortages

To derive an economic lot size formula for optimum production quantity q per cycle of a single product so as to minimize the total average variable cost per unit time, where—

(i) R units per unit time is the uniform demand rate,

(ii) lead time is zero,

(iii) production rate is finite (K units per unit time), $K > R$,

(iv) inventory carrying cost is Rs. C_1 (= IP) per quantity unit per unit time,

(v) shortages are allowed and backlogged.

(vi) shortage cost is Rs. C_2 per quantity unit per unit time.

(vii) set-up cost is Rs. C_3 per set-up.

Solution. The situation, we are going to discuss, is illustrated in *Fig. 20.9.*

Figure shows that there is an inventory cycle. Stocks start at zero and increase for a period t_1.

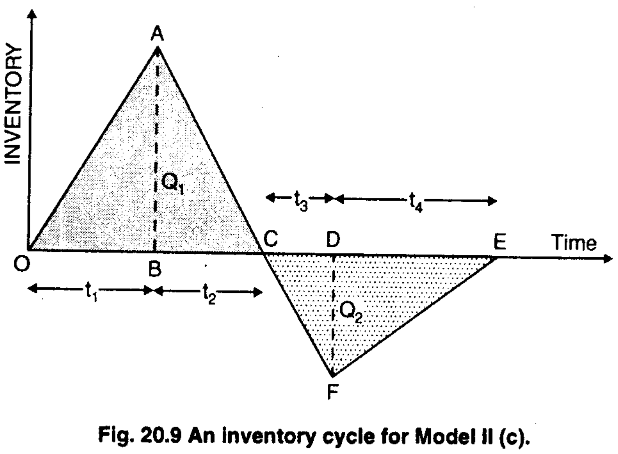


Fig. 20.9 An inventory cycle for Model II (c).

They decline for a period t_2 until they again reach zero at the point where a backlog piles up for the time t_3. At the end of t_3 production starts, and backlog is diminished for the time t_4 when the backlog reaches zero. The cycle then repeats itself after total time $(t_1 + t_2 + t_3 + t_4)$.

Now, holding cost $= C_1 \times \Delta OAC = C_1 \cdot \frac{1}{2} \cdot Q_1 (t_1 + t_2)$

shortage cost $= C_2 \times \Delta EFC = C_2 \cdot \frac{1}{2} \cdot Q_2 (t_3 + t_4)$.

and the set-up cost per set-up is equal to C_3.

Thus, we obtain the average cost per unit time

$$C = \frac{\frac{1}{2}[C_1Q_1 (t_1 + t_2) + C_2Q_2 (t_3 + t_4)] + C_3}{t_1 + t_2 + t_3 + t_4} \qquad ...(20.57)$$

Here C is a function of six variables $(Q_1, Q_2, t_1, t_2, t_3, t_4)$, but there are four relationships which permit us to eliminate all but two variables.

The inventory is zero at O and during the period t_1 an amount Kt_1 is produced but because orders are being filled-up at a rate R, the net increase Q_1 in inventory during t_1 is given by

$$Q_1 = Kt_1 - Rt_1 = t_1 (K - R). \qquad ...(20.58)$$

Now after time t_1, the production is stopped and the stock Q_1 is used up during t_2, and because the rate of use is R, we have

$$Q_1 = Rt_2. \qquad ...(20.59)$$

From (20.58) and (20.59), we have

$$t_1 = \frac{Q_1}{K - R} = \frac{Rt_2}{K - R}. \qquad ...(20.60)$$

During period t_3 shortages accumulate at a rate R, therefore

$$Q_2 = Rt_3. \qquad ...(20.61)$$

During period t_4, production rate is K and demand rate is R, so that the net rate of reduction of shortage becomes $K - R$, and thus we have

$$Q_2 = t_4 (K - R). \qquad ...(20.62)$$

From (20.62) and (20.61), it follows that

$$t_4 = \frac{Q_2}{K - R} = \frac{Rt_3}{K - R}. \qquad ...(20.63)$$

Finally, because the total cycle $(t_1 + t_2 + t_3 + t_4)$ and production q is just sufficient to meet the demand at a rate R, we have

$$q = R (t_1 + t_2 + t_3 + t_4). \qquad ...(20.64a)$$

Now substituting the values of t_1 and t_4 from (20.60) and (20.63) in (20.64a), we get

$$q = R\left(\frac{Rt_2}{K - R} + (t_2 + t_3) + \frac{Rt_3}{K - R}\right) \text{ or } q = \frac{(t_2 + t_3) K}{K - R}. \qquad ...(20.64b)$$

Now eliminating t_1, t_4, Q_1 and Q_2 from (20.57) by using relations (20.60) and (20.63), we get

$$C = \frac{\frac{1}{2}\left\{C_1 (Rt_2)\left(\frac{Rt_2}{K - R} + t_2\right) + C_2Rt_3\left(t_3 + \frac{Rt_3}{K - R}\right)\right\} + C_3}{\frac{Rt_2}{K - R} + t_2 + t_3 + \frac{Rt_3}{K - R}}$$

$$= \frac{\frac{1}{2}\left\{\frac{C_1t_2^2RK}{K - R} + \frac{C_2t_3^2RK}{K - R}\right\} + C_3}{(t_2 + t_3)\left(1 + \frac{R}{K - R}\right)}$$

$$= \frac{\frac{1}{2}[C_1t_2^2 + C_2t_3^2]\frac{RK}{K - R} + C_3}{(t_2 + t_3)\left(\frac{K}{K - R}\right)} = \frac{\frac{1}{2}[C_1t_2^2 + C_2t_3^2] RK + C_3 (K - R)}{K (t_2 + t_3)}$$

Thus, cost equation (20.57) now becomes

$$C (t_2, t_3) = \frac{\frac{1}{2}[C_1t_2^2 + C_2t_3^2] RK + C_3 (K - R)}{K (t_2 + t_3)} \qquad ...(20.65)$$

To find the best values t_2^* and t_3^* of t_2 and t_3, we differentiate partially w.r.t. 't_2' and 't_3' and set the results equal to zero. The resulting equations are then solved. We omit the details and only state the results as follows :

$$t_2^* = \sqrt{\left(\frac{2C_3C_2\,(1 - R/K)}{R\,(C_1 + C_2)\,C_1}\right)} \quad \text{...(20.66)}$$

$$t_3^* = \sqrt{\left(\frac{2C_3C_1\,(1 - R/K)}{R\,(C_1 + C_2)\,C_2}\right)} \quad \text{...(20.67)}$$

and from (20.64b) and (20.62), we obtain the optimum order quantity (again omitting the details),

$$q^* = \sqrt{\left(\frac{2RC_3\,(C_1 + C_2)}{C_1C_2}\right)} \cdot \sqrt{\left(\frac{1}{1 - R/K}\right)} \quad \text{...(20.68)}$$

$$= \sqrt{\left(\frac{2RC_1C_3\,(1 - R/K)}{(C_1 + C_2)\,C_2}\right)} \quad \text{...(20.69)}$$

Finally, the minimum cost is given by

$$C^* = \sqrt{\left(\frac{2RC_1C_2C_3\,(1 - R/K)}{C_1 + C_2}\right)} \cdot \quad \text{...(20.70)}$$

Q. Find the optimum production quantity of a single product for a generalized economic lot size model so as to minimize the total average variable cost per unit time. using the following information :
(i) $\lambda \to$demand rate, (ii) $\psi \to$ production rate ($\psi > \lambda$), (iii) $I_c \to$ inventory holding cost per unit per unit time,
(iv) $\pi \to$ shortage cost (blacklogging) per unit time, (v) $A \to$ setup cost per setup, (vi) lead time is zero.
[**Hint.** Change the notations in ***Model II (c).***] **[Delhi MA/M.Sc. (Stat.) 95, M.Sc. (OR) 90]**

Some special cases duduced from *Model II (c)* :

Case 1. If in ***Model II (c),*** we assume that the production is instantaneous, *i.e.* $K \to \infty$, the corresponding results thus obtained from above will become as follows :

$$C\,(t_2\,,\,t_3) = \frac{\tfrac{1}{2} R\,[C_1t_2^2 + C_2t_3^2] + C_3}{t_2 + t_3} \qquad \textbf{(Cost Equation)}$$

Equations (20.68), (20,69) and (20.70) become

$$q^* = \sqrt{\left(\frac{2RC_3\,(C_1 + C_2)}{C_1C_2}\right)}, \quad Q^* = \sqrt{\left(\frac{2RC_1C_3}{(C_1 + C_2)\,C_2}\right)},$$

$$C^* = \sqrt{\left(\frac{2RC_1C_2C_3}{C_1 + C_2}\right)}, \text{respectively.}$$

Case 2. In this case, if shortages are also not permitted, *i.e.* $C_2 \to \infty$, $t_3 = 0$, we can obtain the same results which were obtained in ***Model I (a).*** These results are as follows :

$$C\,(t_2) = \tfrac{1}{2} RC_1t_2 + C_3/t_2 = \sqrt{2RC_1C_3}\,, \quad q^* = \sqrt{\left(\frac{2RC_3}{C_1}\right)}, \quad Q^* = 0$$

20.14-4. Illustrative Examples

Example 24. *The demand for an item in a company is 18,000 units per year, and the company can produce the item at a rate of 3,000 per month. The cost of one set-up is Rs. 500.00 and the holding cost of one unit per month is 15 paise. The shortage cost of one unit is Rs. 20.00 per year. Determine the optimum manufacturing quantity and the number of shortages. Also, determine the manufacturing time and the time between set-ups.*

Solution. Here we are given that : C_1 = Re. 0.15, C_2 = Rs. 20, C_3 = Rs. 500, K = 3,000 units/months, R = 18,000/12 units/month = 1500 units/month.

$$\therefore \quad q^* = \sqrt{\left(\frac{2C_3R\,(C_1 + C_2)}{C_1C_2}\right)} \cdot \sqrt{\left(\frac{K}{K - R}\right)}$$

$$= \sqrt{\left(\frac{2 \times 500 \times 1500 \times (0.15 \times 12 + 20)}{(12 \times 0.15) \times 20}\right)} \sqrt{\left(\frac{3{,}000}{3{,}000 - 1{,}500}\right)} = 4{,}670 \text{ units.} \quad \textbf{Ans.}$$

Number of shortages are given by,

$$S = \frac{C_1}{C_1 + C_2} q^* \left(1 - \frac{R}{K}\right) = \frac{0.15 \times 12}{0.15 \times 12 + 20} \times 4{,}670 \left(1 - \frac{1500}{3000}\right) = \frac{1.8}{21.8} \times 4{,}670 \times 0.5 = 193 \text{ units.} \quad \textbf{Ans.}$$

$$\text{Manufacturing time} = \frac{q^*}{K} = \frac{4{,}670}{3000 \times 12} = 0.13 \text{ year.} \quad \textbf{Ans.}$$

$$\text{Time between setups} = \frac{q^*}{R} = \frac{4{,}760}{18{,}000} = 0.26 \text{ year.} \quad \textbf{Ans.}$$

EXAMINATION PROBLEMS (On Model II)

1. A manufacturer has to supply his customers 24,000 units of his product per year. The demand is fixed and known. The customer has no storage space and so the manufacturer has to ship a day's supply each day. If the manufacturer fails to supply, the penalty is Re. 0.20 per unit per month. The inventory holding cost amounts to Re. 0.10 per unit per month and the setup cost is Rs. 350 per production run. Find the optimum lot size for the manufacturer.
[**Hint.** R = 24,000/year = 2000/month, C_1 = Re. 0.10, C_2 = Re. 0.20, C_3 = 350. Use formula (2.54).]
[**Ans.** 4744 units per run.]

2. The demand for a certain item is 16 units per period. Unsatisfied demand causes a shortage cost of Re. 0.75 per unit per short period. The cost of initiating purchasing action is Re. 1 per purchase and the holding cost is 15% of average inventory valuation per period. Item cost is Rs. 8.00 per unit. (Assume that shortages are being back-ordered at the above mentioned cost). Find the minimum cost of purchase quantity.
[**Hint.** R = 16, C_1 = 0.15 × 8, C_2 = 0.75, C_3 = 15.00. Use formula (2.54) and (2.55).]
[**Ans.** q^* = 33.3 units; C_{min} = Rs. 14.88 (nearly)]

3. A contractor undertakes to supply diesel engines to a truck manufacturer at a rate of 25 per day. He finds that the cost of holding a completed engine in stock is Rs. 16 per month, and there is a clause in the contract penalizing him Rs. 10 per engine per day late for missing the scheduled delivery date. Production of engines is in batches, and each time a new batch is started there are setup costs of Rs. 10,000. How frequently should batches be started, and what should be the initial inventory level at the time each batch is completed. **[Agra 98; JNTU 98]**
[**Hint.** C_1 = 16, C_2 = 10, C_3 = 10,000, R = 25 engines per day.]
[**Ans.** q^* = 943 engines (approx.), t^* = q^*/38 dyas.]

4. A commodity is to be supplied at a constant rate of 200 units per day. Supplies for any amounts can be had at any required time, but each ordering costs Rs. 50.00; costs of holding the commodity in inventory is Rs. 2.00 per unit per day while the delay in the supply of the item induces a penalty of Rs. 10.00 per unit per dalay of 1 day.
Formulate the average cost function of this situation and find the optimal policy (q, t), where t is the re-order cycle period and q is the inventory level after re-order. What should be the best policy if the penalty cost becomes infinite.
[**Hint.** R = 200, C_3 = 50, C_1 = 2, C_2 = 10; when $C_2 \to \infty$, $q^* = \sqrt{(2C_3R/C_1)}$.] **[Mysore B.E. (Mech.) 82]**
[**Ans.** q^* = 109.5 units, t^* = q^*/R = 1/2 day. $C_2 \to \infty$, q^* = 100 units. t^* = q^*/R = 1/2 day.]

5. The demand for an item is 18,000 units per year. The holding ocst is Rs. 1.20 per unit time and the cost of shortage is Rs. 5.00, the production cost is Rs. 400.00. Assuming that replenishment rate is instantaneous, determine the optimal order quantity. **[Bombay (B.Sc. Stat.) 76]**
[**Hint.** R = 18,000, C_1 = 1.20, C_2 = 5,00, C_3 = 400.00.]
[**Ans.** q^* = 3,860 units, t^* = q^*/R = 0.215 year, $n = R/q^*$ = 18,000/3,860 = 4.66 orders per year.]

6. In a certain manufacturing company, the annual requirement is 24,000 units, the supply is instantaneous and shortage permits, the cost of order each time is Rs. 350. The cost of carrying inventory is Re. 0.10 per unit per month, the cost of shortage is Re. 0.20 per unit per month. Find the economic quantities to be procured and the total cost of inventory in such case. **[Ranchi M.Sc. (Stat.) 82; Jodhpur B.E. (Mech.) 80]**
[**Ans.** (i) q^* = 3600, (ii) 50 orders per year, (iii) C_{min} = Rs. 6000.]

7. A certain product has demand of 25 units per month and the items are withdrawn uniformly. Each time a production run is made the setup cost is Rs. 15. The production cost is Re. 1 per item and inventory holding cost is Re. 0.30 per item per month.
   (i) Assuming shortages are not permitted, determine how often to make a production run and what size it should be ? **[Meerut 85]**
   (ii) If shortage cost is Rs. 1.50 per item per month, determine how often to make a production run and what size it should be ? **[Bangalore M.E. (Mech.) 82]**
[**Ans.** (i) q^* = 50 units, (ii) q^* = 54.7 units.]

8. (i) The price of an item of inventory has increased from Rs. 10 per unit last year to Rs. 20 per unit this year. The annual usage and other relevant costs remain unchanged. If the EOQ for this item last year was determined as 1000 units, what should be the EOQ this year ?

   (ii) If the annual demand for a product is D units, ordering (buying) cost is B, carrying cost per unit per year is C, and the back-ordering (shortage) cost per unit per year is S, indicate the total relevant costs of the inventory system (No proof is needed).

9. A dealer supplies you the following information with regard to a product dealt with by him :

   Annual demand = 5,000 units, Buying cost = Rs. 250/– per order, Inventory carrying cost = 30% per year, Price = Rs. 100 per unit.

   The dealer is considering the possibility of allowing some back-orders to occur for the product. He has estimated that the annual cost of back-ordering (allowing shortage) of the product that will be Rs. 10/– per unit.

   (i) What quantity of the product should he allow to be back-ordered ?

   (ii) What should be the optimum number of units of the product he should buy in one lot ?

   (iii) How much additional cost will he have to incur on inventory if he does not permit back-ordering ?

   [**Ans.** (i) 577.35, (ii) 433 units, (iii) Rs. 4330.13]

10. A dealer supplies you the following information with regard to a product dealt with by him :

    Annual demand = 10,000 units, Ordering cost = Rs. 10 per order, Inventory carrying cost = 20% of the value of the inventory per year, Price = Rs. 20 per unit.

    The dealer is considering the possibility of allowing some back-order (stock-out) to occur. He has estimated that the annual cost of back-ordering will be 25% of the value of inventory.

    (i) What should be the optimum number of units of the product he should buy in one lot ?

    (ii) What quantity of the product should he allow to be back-ordered, if any ?

    (iii) What will be the maximum quatnity of inventory at any time of the year ?

    (iv) Would you recommend to him to allow back-ordering ? If so, what would be the annual cost saving by adopting the policy of back-ordering ?

20.15. MULTI-ITEM DETERMINISTIC MODELS (*THE EOQ WITH CONSTRAINTS*)

20.15–1. Model III : Multi-item with One Constraint

So far, we have consider each item separately. But, if there exists a relationship among the items under some limitations, then it is not possible to consider them separately. After constructing the cost equation in such models, we use the method of Lagrange's multiplier to minimize the cost in simple cases. While dealing with all such problems, we first solve the problem without considering the effect of limitations.

We consider the problems with the following assumptions :

(i) n items with instantaneous production and no lead time.

(ii) R_i is the uniform demand rate for the ith item ($i = 1, 2, \ldots, n$).

(iii) $C_1^{(i)}$ is the holding (or carrying) cost per unit of the quantity of ith item.

(iv) shortages are not allowed (i.e. $C_2^{(i)} = 0$).

(v) $C_3^{(i)}$ is the set-up cost per production run for the ith item.

(vi) q_i is the total quantity of the ith item produced at the beginning of the production run.

Now proceeding exactly as in ***Model I (a)***, we get the cost per unit time for the *i*th item as :

$$c_i(t) = \tfrac{1}{2} C_1^{(i)} R_i t + C_3^{(i)}/t \quad \text{or} \quad c_i(q_i) = \tfrac{1}{2} C_1^{(i)} q_i + C_3^{(i)} R_i/q_i \qquad \ldots(20.71)$$

[see eqns. (20.12) and (20.17) of ***Model I (a)***]

Hence summing up these costs for $i = 1, 2, \ldots, n$, we get

$$C = \sum_{i=1}^{n} [\tfrac{1}{2} C_1^{(i)} q_i + C_3^{(i)} R_i/q_i] \quad \textbf{(Cost-equation)} \qquad \ldots(20.72)$$

To determine the optimum value of q_i ($i = 1, 2, \ldots, n$) so that the total cost C is minimum, we have necessary condition $\partial C/\partial q_i = 0$. Therefore, we get

$$\frac{\partial C}{\partial q_i} = \tfrac{1}{2} C_1^{(i)} - C_3^{(i)} R_i/q_i^2 = 0$$

which gives

$$q_i = \sqrt{(2C_3^{(i)} R_i/C_1^{(i)})}$$

Since $\partial^2 C/\partial q_i^2 > 0$ for all q_i, the total cost C is minimum. Hence the optimum value of q_i is given by

$$q_i^* = \sqrt{(2C_3^{(i)} R_i / C_1^{(i)})}\ , \ i = 1, 2, \ldots, n. \qquad \ldots(20.73)$$

We now proceed to consider the effect of limitations, *viz.* (i) *limitation on investment,* (ii) *limitation on stocked units,* and (iii) *limitation on warehouse floor space.*

20.15–2. Model III (a) : Limitation on Investment

In this case, there is an upper limit, say M (in Rs.) on the amount to be invested on inventory.

Let $C_4^{(i)}$ be the unit price of ith item, Then

$$\sum_{i=1}^{n} C_4^{(i)} q_i \leq M. \qquad \ldots(20.74)$$

Now our problem is to minimize the total cost C given by the eqn. (20.72) subject to the additional constraint (20.74) above. In this situation, two cases may airse :

Case 1. ***When*** $\sum_{i=1}^{n} C_4^{(i)} q_i^* \leq M$, q_i^* ***given by* (20.73).**

In this case, there is no difficulty, q_i^* given by (20.73), is the required optimal value of q_i.

Case 2. ***When*** $\sum_{i=1}^{n} C_4^{(i)} q_i^* > M$, q_i^* ***given by* (20.73)**

In this case, q_i^* ($i = 1, 2, \ldots n$) given by (20.73) are not the required optimal values. Therefore, we shall use the *Lagrange's multiplier technique* as follows :

Let us formulate the Lagrangian function :

$$L = \sum_{i=1}^{n} \left(\frac{1}{2} C_1^{(i)} q_i + \frac{C_3^{(i)} R_i}{q_i} \right) + \lambda \left(\sum_{i=1}^{n} C_4^{(i)} q_i - M \right)$$

with the help of (20.72) and (20.74). Here λ is a Lagrange's multiplier.

The necessary condition for L to be minimum is $\dfrac{\partial L}{\partial q_i} = \dfrac{\partial L}{\partial \lambda} = 0$ ($i = 1, 2, \ldots, n$). Therefore,

$$\frac{\partial L}{\partial q_i} = \frac{1}{2} C_1^{(i)} - \frac{C_3^{(i)} R_i}{q_i^2} + \lambda\, C_4^{(i)} = 0, \ (i = 1, 2, \ldots, n) \qquad \ldots(20.75)$$

and

$$\frac{\partial L}{\partial \lambda} = \sum_{i=1}^{n} C_4^{(i)} q_i - M = 0. \qquad ..(20.76)$$

These equations give

$$q_i^* = \sqrt{\left(\frac{2C_3^{(i)} R_i}{C_1^{(i)} + 2\lambda^* C_4^{(i)}} \right)} \qquad \ldots(20.77)$$

and

$$\sum_{i=1}^{n} C_4^{(i)} q_i^* = M. \qquad \ldots(20.78)$$

The second equation implies that q_i^* must satisfy the invenstment constraint in equality sense.

Since q_i^* depends on λ^* (the optimal value of λ), λ^* can be found by systematic trial and error. By trying successive positive values of λ, the value of λ^* should result in simultaneous value of q_i^* satisfying the given constraint in equality sense. Thus determination of λ^* will automatically determine q_i^*.

The following interesting example will make the procedure clear :

Example 25. *Consider a shop which produces three items. The items are produced in lots. The demand rate for each item is constant and can be assumed to be deterministic. No back orders are to be allowed.*

The pertinent data for the items is given in the following table :

Item	1	2	3
Holding cost (Rs.)	20	20	20
Set-up cost (Rs.)	50	40	60
Cost per unit (Rs.)	6	7	5
Yearly demand rate	10,000	12,000	7,500

Determine approximately the Economic Order Quantities when the total value of average inventory levels of three items is Rs. 1000. **[Agra 98; Rohilkhand M.Sc. (Math.) 93; Kanpur M.Sc. (Math.) 93]**

Solution. First of all, we compute the optimal value q^* without considering the effect of restriction by using the formula (20.73). Thus, we get

$$q_1^* = \sqrt{\left(\frac{2 \times 50 \times 10{,}000}{20}\right)},\quad q_2^* = \sqrt{\left(\frac{2 \times 40 \times 12{,}000}{20}\right)},\quad q_3^* = \sqrt{\left(\frac{2 \times 60 \times 7{,}500}{20}\right)}$$

$$= 100\sqrt{5} = 223 \text{ approx.} \qquad = 40\sqrt{30} = 216 \text{ approx.} \qquad = 150\sqrt{2} = 210 \text{ approx.}$$

Since the average optimal inventory at any time is $\frac{1}{2} q_i^*$, the investment over the average inventory is obtained by replacing q_i by $\frac{1}{2} q_i^*$ in (20.78)

$$= \sum_{i=1}^{n} C_4^{(i)} \,(1/2\, q_i^*) = \text{Rs.} \,(\, 6 \times 223/2 + 7 \times 216/2 + 5 \times 210/2\,) = \text{Rs. } 1950.00.$$

We observe that the amount of Rs. 1950 is greater than the upper limit of Rs. 1000. Therefore, we try to find the suitable value of λ by trial and error method for computing q_i^* by using the formula (20.77).

If we set $\lambda = 4$ in (20.77), we get

$$q_1^* = \sqrt{\left(\frac{2 \times 50 \times 10{,}000}{20 + 2 \times 4 \times 6}\right)} = 121,\ q_2^* = \sqrt{\left(\frac{2 \times 40 \times 12{,}000}{20 + 2 \times 4 \times 7}\right)} = 112,\ q_3^* = \sqrt{\left(\frac{2 \times 60 \times 7{,}500}{20 + 2 \times 4 \times 5}\right)} = 123,$$

and hence the cost of average inventory $= 6 \times \frac{121}{2} + 7 \times \frac{112}{2} + 5 \times \frac{123}{2} =$ Rs. 1112.50,

which is also greater than Rs. 1000.

Again, if we set $\lambda = 5$ in (20.77), we shall obtain $q_1^* = 111$, $q_2^* = 102$, $q_3^* = 113$.
and the corresponding cost of average inventory is Rs. 972.50 which is less than Rs. 1000.

From this, we conclude that the most suitable value of λ lies between 4 and 5.

To find the most suitable value of λ, we draw a graph between cost of average inventory and the value of λ as shown in *Fig. 20.10*.

This graph indicates that $\lambda = 4.7$ (approx.) is the most suitable value corresponding to which the cost of inventory is Rs.999.50 which is sufficiently close to Rs. 1000.

For $\lambda = 4.7$, we obtain the required optimal values of three items as :

$q_1^* = 114$, $q_2^* = 105$, and $q_3^* = 116$. **Ans.**

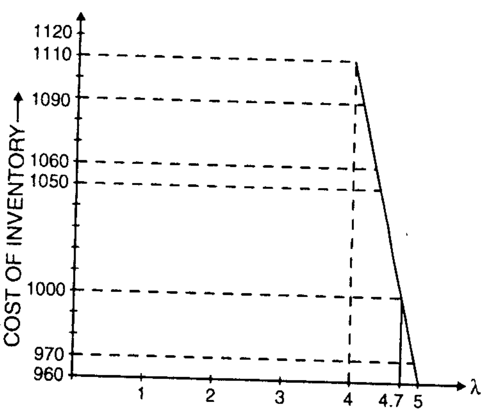


Fig. 20.10

20.15-3. Model III (b) : Limitation on Inventories

In this case, the upper limit of average number of all units in the stock is (say), N, *i.e.*, number of all units in inventory should not exceed N for all items. Since the average number of units at any time is $\frac{1}{2} q_i$, we have to minimize the cost C given by eqn. (20.72), subject to the condition

$$\frac{1}{2} \cdot \sum_{i=1}^{n} q_i \le N. \qquad \ldots(20.79)$$

Now two possibilities may arise :

Case. 1. If $\frac{1}{2}\sum_{i=1}^{n} q_i^* \le N$, there is no difficulty and the optimum values are q_i^* $(i = 1, 2, \ldots, n)$ given by the formula (20.73).

Case 2. If $\frac{1}{2}\sum_{i=1}^{n} q_i > N$, then the optimal values given by (20.73) are not the required values. So we use Largrange's multiplier technique.

For this, we construct the Lagrangian function as

$$L = \sum_{i=1}^{n}\left(\frac{1}{2}C_1^{(i)}q_i + \frac{C_3^{(i)}R_i}{q_i}\right) + \lambda\left(\frac{1}{2}\sum_{i=1}^{n} q_i - N\right)$$

where $\lambda > 0$ is the Lagrangian multiplier.

The optimum values of q_i are obtained by setting

$$\frac{\partial L}{\partial q_i} = \frac{1}{2}C_1^{(i)} - \frac{C_3^{(i)}R_i}{q_i^2} + \frac{\lambda}{2} = 0, \text{ and } \frac{\partial L}{\partial \lambda} = \frac{1}{2}\sum_{i=1}^{n} q_i - N = 0. \qquad \text{for } i = 1, 2, \ldots, n$$

Solving these two equations, we get

$$q_i^* = \sqrt{\left(\frac{2C_3^{(i)}R_i}{C_1^{(i)} + \lambda*}\right)}, \; i = 1, 2, \ldots, n \qquad \ldots(20.80)$$

and

$$\sum_{i=1}^{n} q_i^* = 2N. \qquad \ldots(20.81)$$

To obtain the values of q_i^* from (20.80), we find the optimal value $\lambda*$ of λ by successive trail and error method, subject to the condition given by (20.81).

The following example will illustrate the procedure.

Example 26. *A company producing three items has a limited storage space of averagely 750 items of all types. Determine the optimal production quantities for each item separately, when the following information is given :*

Product :		1	2	3
Holding Cost (Rs.)	:	0.05	0.02	0.04
Set-up Cost (Rs.)	:	50	40	60
Demand rate (per unit)	:	100	120	75

Solution. Neglecting the restriction of the total value for inventory level, we find

$$q_1^* = \sqrt{\left(\frac{2 \times 50 \times 100}{0.05}\right)} \qquad q_2^* = \sqrt{\left(\frac{2 \times 40 \times 120}{0.02}\right)} \qquad q_3^* = \sqrt{\left(\frac{2 \times 60 \times 75}{0.04}\right)}$$

$$= 100\sqrt{20} = 447 \text{ (nearly)} \qquad = 100\sqrt{48} = 693 \text{ (nearly)} \qquad = 100\sqrt{21.5} = 464 \text{ (nearly)}$$

Therefore, the total average inventory at time is = ½ (447 + 693 + 464) = 802 units.

But, we are given the storage capactity for 750 items per year and, therefore, we have to find the value of parameter λ by trial and error substitution process.

From this, we observe that the average inventory level is less than the available amount of items. So we should try for some other values of $\lambda = 0.004, 0.003, 0.002$, etc.

We now try for λ = 0.002. For $\lambda = 0{\cdot}002$, we get $q_1^* = 428$, $q_2^* = 628$, $q_3^* = 444$.

$\therefore$ Average inventory level becomes = ½ [428 + 628 + 444] = 750 units,
which is equivalent to the given amount of average inventory level.

Hence, the optimal production quantities for each of the three items are :

$$q_1^* = 428 \text{ units}, \quad q_2^* = 628 \text{ units}, \text{ and } q_3^* = 444 \text{ units}. \quad \textbf{Ans.}$$

20.15–4. Model III (c) : Limitation on Floor Space (Storage Space)

In this model, the inventory system includes n (> 1) items which are competing for a limited *storage space*. An interaction between the different items occuring due to this limitation can be included as an additional constraint.

Let

$A =$ the maximum storage area (in sq. meter) available for the n items.

$a_i =$ storage area required per unit of the ith item, and $q_i =$ the amount ordered of the ith item.

Thus, the storage requirement constraint becomes :

$$\sum_{i=1}^{n} a_i \; q_i \le A \, , q_i > 0. \qquad ...(20.82)$$

The relevant inventory costs for each item should be the same as in the case of ***Model I.*** Thus, our problem becomes :

$$\text{Minimize. } C = \sum_{i=1}^{n} \left(\frac{1}{2} C_1^{(i)} q_i + \frac{C_3^{(i)} R_i}{q_i} \right), \text{ subject to the constraint (20.82).}$$

The Lagrange's multiplier method yields the general solution of this problem. However, before applying this method, it is necessary to check whether the unconstrained value of q_i given by (20.78) satisfy the storage constraint. If not, the new optimal values of q_i must be determined which will satisfy the storage constraint in equality sense. This is done by first formulating the Lagrangian function :

$$L = \sum_{i=1}^{n} \left(\frac{1}{2} C_1^{(i)} q_i + \frac{C_3^{(i)} R_i}{q_i} \right) + \lambda \left(\sum_{i=1}^{n} a_i q_i - A \right),$$

where $\lambda > 0$ is the *Lagrange's* multiplier.

Proceeding as in ***Model III (a)***, we obtain the optimal value

$$q_i^* = \sqrt{\left(\frac{2C_3^{(i)} R_i}{C_1^{(i)} + 2\lambda^* a_i} \right)} \, , i = 1, 2, \dots , n \qquad ...(20.83)$$

and

$$\sum_{i=1}^{n} a_i q_i^* = A. \qquad ...(20.84)$$

The second equation implies that q_i^* must satisfy the storage constraint in equality sense. The determination of λ^* by usual *trial and error method* automatically yield the optimum value q_i^*.

We illustrate this model by an example.

Q. Discuss a deterministic inventory system with multiple items and limited floor space. **[Delhi MA/M.Sc. (Stat.) 95]**

Example 27. *Consider the inventory problem with three items (i.e. $n = 3$). The parameters of the problem are shown in the table below.*

Item (i)	R_i (units)	$C_3^{(i)}$ (Rs.)	$C_1^{(i)}$ (Rs.)	a_i (mt^2)
1	20	100	30	1
2	40	50	10	1
3	30	150	20	1

Assume that the total available storage area is given by $A = 25\ mt^2$. Find the optimal order quantities of three items.

Solution. Substituting the given values in the formula (20.83), we construct the following table :

λ	q_1	q_2	q_3	$\sum_{i=1}^{3} a_i q_i - A$
0	11.5	20.0	21.2	+ 27.7
5	10.0	14.1	17.3	+ 16.4
10	9.0	11.5	14.9	+ 10.4
15	8.2	10.2	13.4	+ 6.6
20	7.6	8.9	12.2	+ 3.7
25	7.1	8.2	11.3	+ 1.6
30	6.7	7.6	10.6	− 0.1

We observe that for $A = 25\ mt^2$, the storage constraint is satisfied in equality sense for some value of λ between 25 and 30. This value is equal to λ^* which may be estimated by drawing a graph (as in *Fig. 20.10*). The corresponding values of q_i give us q_i^* directly. As seen from the table, the values of λ^* must be very near to 30 and thus the optimal values q_i^* are approximately given by $q_1^* = 6.7$, $q_2^* = 7.6$, $q_3^* = 11.6$.

Remark. If $A \geq 52.4$ in this example, then the unconstrained value of q_i (corresponding to $\lambda = 0$) should yield q_i^*. In this case, the constraint should be neglected.

EXAMINATION PROBLEMS (On Model III)

1. Consider a shop which produces and stocks three items. The management desires never to have an investment in inventory of more than Rs. 15000. The items are produced in lots. The demand rate for each item is constant and can be assumed to be deterministic. No back orders are to be allowed. The pertinent data for the items are given in the following table. The carrying charge on each item is 20% of average inventory valuation per annum. Determine the optimal lot size for each item.

Data for Problem

Item	1	2	3
Demand rate (units/year)	1,000	500	2,000
Variable cost (Rs. per unit)	20	100	50
Set-up cost per lot (Rs.)	50	75	100

[**Hint.** Proceed as in solved example for **Model III (a)**.]

2. The following table gives the sales/year, rate of production, inventory carrying cost and set-up cost for 5 items manufactured by M/s Bolts and Nuts Ltd. using the same equipment.

Product	Sales/year	Production/day	Inventory cost per unit per yr.	Set-up cost
1	5,00,000	10,000	0.020	8.00
2	13,000	1,000	0.005	10.00
3	2,00,000	10,000	0.008	17.00
4	1,80,000	2,000	0.022	5.00
5	95,000	1,000	0.015	6.80

Obtain the optimal number of cycles in a year assuming that there are 300 working days available in a year.
[**Hint.** Proceed as solved example of ***Model III (a)***.]

3. A small shop produces three machine parts 1, 2, 3 in lots. The shop has only 6.50 sq. ft. storage space. The appropriate data for three items are presented in the following table :

Item	1	2	3
Demand rate (units/year)	5,000	2,000	10,000
Procurement cost (Rs.)	100	200	75
Cost per unit (Rs.)	10	15	5
Floor space required (sq. ft./unit)	0.70	0.80	0.40

The carrying charge on each item is 20% of average inventory valuation per annum. If no stock-out are allowed, determine the optimal lot size for each item.
[**Hint.** Proceed as solved example of **Model III (c)**.]

4. (i) The uniform annual demands for two bulky items are 90 tons and 160 tons, respectively. The carrying costs are Rs. 250 and Rs. 200 per ton per year, and set-up costs Rs. 50 and Rs. 40 per production, respectively. No shortages are allowed. Space considerations restrict the average amount inventory of items to 4000 ft^3. A ton of the first item occupies 1000 ft^3, and a ton of second item 500 ft^3. Find the optimal lot-size. **[Meerut (Maths.) 97 P]**
[**Hint.** Proceed as solved example of **Model III (c)**. **Ans.** $q_1^* = 2$ tons, $q_2^* = 12$ tons.]
(ii) Solve this problem when shortages are now allowed. **[Meerut 97 P]**

5. Four products A, B, C and D are to be manufactured successively in batches on the same machine. The consumption and production rates, batch set-up and item holding costs for each product are shown in the following table :

Product	Consumption per year	Production rate per day	Holding cost per item per day (Rs.)	Set-up cost (Rs.)
A	10,000	250	0.85	42
B	5,000	100	1.20	18
C	8,000	200	2.00	60
D	12,000	300	0.65	20

If there are 250 working days per year, calculate the production batch sizes for each product and how many complete production run of all four products should be made per year.

II-Dynamic or Fluctuating Demand Models (IV & V)

Now we shall discuss inventory models in which demand cannot be completely pre-determined. It is uncertain and can fluctuate on either way. In several practical situations, it is found that neither the consumption rate (R) nor the lead time (L) is constant throughout the year. So, in order to deal with such uncertainties in consumption rate and lead time, an extra stock is maintained (preserved) to meet out the demands, if any. This extra stock is termed as ***"Buffer Stock"*** or ***"Safety Stock"*** usually denoted by 'B'.

In these models we want to discuss a method to determine the Re-Order Level (R.O.L.), that is, the time when we should place an order by taking into consideration the interval between placing the order and receiving the supply. For example, we would like to place a new order precisely at the time when inventory level falls to zero. These models are classified as ***Model IV*** and ***Model V.***

20.16. MODEL IV : TO DETERMINE RE-ORDER LEVEL AND OPTIMUM BUFFER STOCK

To determine the buffer stock, we approximate the estimated *maximum lead time* and *normal lead* time for a particular item.

Let L_d = difference between maximum and normal lead times, R = consumption rate during the lead time.

Then, the buffer stock is given by the formula, $B = L_d \times R$. That is,

Buffer stock = (Max. lead time – normal lead time) × consumption rate

For example, let the monthly consumption for a certain item be 300 units, the normal lead time be 15 days, and the maximum lead time is estimated as 3 months. Then, the buffer stock is given by

$$B = (3 - ½) \times 300 = 750 \text{ units.}$$

Now *three situations* may arise :

(i) If we do not maintain a buffer stock, then the total requirements for inventory during the lead time will become $L_d R$. This implies that as soon as the stock reaches a level $L_d R$, we place an order for q units. We call this the *'recorder point'* or *'reorder level'*, which is given by

$$\text{R.O.L.} = L_d R \qquad ...(20.85)$$

This relationship is well illustrated in *Fig. 20.11.* This policy results in shortages for about half the time.

(ii) In order to avoid the shortage, we have to maintain a buffer stock 'B' and places an order when stock level reaches $B + L_d R$. Hence the reorder level is given by

$$\text{R.O.L.} = B + L_d R \qquad ...(20.86)$$

For example, if the *monthly consumption rate* for items is 150 units, the *normal lead time* is 15 days, and the *buffer stock* is 200 units, then

$$\text{ROL} = 200 + ½ \,.\, 150 = 275 \text{ units.}$$

(iii) If we take t days for reviewing the reorder system, then on the assumption of uniform consumption rate, we get

$$\text{ROL} = B + L_d R + Rt/2. \qquad ...(20.87)$$

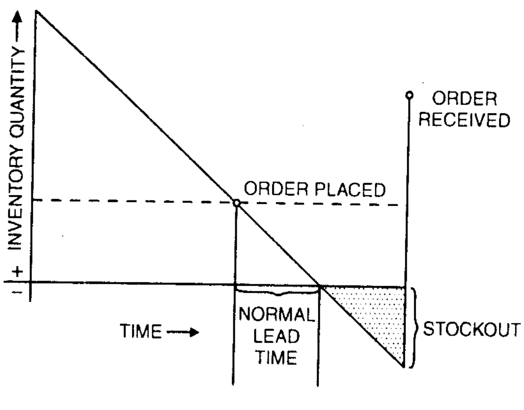


Fig. 20.11 Inventory level with no safety stock, resulting in shortage.

Optimum Buffer Stock. When the buffer stock maintained is very low, naturally the inventory holding cost would be low but the shortages will occur very frequently and the cost of shortages would be very high. On the other hand, if the buffer stock maintained is rather large, shortages would be rather rare, resulting into low shortage costs but the inventory costs would become high. So it is necessary to adjust a balance (compromise) between the cost of shortages and cost of inventory holding to arrive at the *optimum buffer stock.*

Q. Define Re order level. **[Bhubnashwar (IT) 2004]**

The following example will illustrate the procedure.

Example 28. *A small camera maker sells imported electronic flash gun with his camera, as an optional accessory. Last 3 month's records indicate that the average demand for the flash guns was about 100 units per month, the actual demand varying generally between 70 and 140 units per month. Only thrice had the demand exceeded 140 and was 150, 160 and 180 per month. The camera maker, by an agreement with a reliable overseas suppliers, receives 100 guns each month. Calculate the most economic buffer stock the supplier should hold. Assume inventory carrying charges of 20% and the landed cost of gun Rs. 200 per unit. In case of excess demand camera maker purchases extra units from other importers at a premium of Rs. 50 per unit.*

Solution. Let the buffer stock kept by camera-man be of 60 units. We are given that :

q = 100 units per month, P = Rs. 200.00 per unit

I = Re. 0.20 per unit (inventory carrying charges), C_2 = Rs. 50.00 per unit.

Thus, average inventory = (Buffer stock + ½ q) , = (60 + ½ × 100)= 110 units.

Total average inventory costs for a period of 3 months. = Rs. (110 × 200 × 0.20) × 3 = Rs. 13,200.00.

Obviously, the shortage of 20 units would occur only once in 3 months because starting stock is 160 (= 100 + 60) and consumption rate is 180 per month.

Therefore, total shortage cost = Rs. 20 × 50 = Rs. 1,000.

Thus, total cost = Holding cost + Shortage cost. = Rs. (13,200 + 1,000) = Rs. 14,200.

But, the camera-man would not necessarily like to keep a buffer stock of 60 units. If possible, he would like to have a lesser buffer stock.

To find the optimum buffer stock, we compute the total costs in the same way by considering the less buffer stock. These computations are shown in the following table.

Buffer Stock (units)	*Max. Stock (Buffer + q)*	***Inventory costs for** 3 **months (Rs.)***	***Shortage costs for** 3 **months (Rs.)***	***Total costs for** 3 **months (Rs.)***
60	160	13,200	1,000	14,200
55	155	12,600	1,500	14,100
→ 50	150	12,000	2,000	14,000
45	145	11,400	2,750	14,150

From this table we observe that when the amount of buffer stock decrease from 60 to 50, the total cost also decreases from Rs. 14,200 to Rs. 14,000. On further decreasing the buffer stock from 50 to 45, the total cost starts increasing. Hence the optimum buffer stock is around 50 units.

EXAMINATION PROBLEM

Q. A contractor undertakes to supply diesel engines to a truck manufacturer at a rate of 25 engines per day. He finds that cost of holding an engine in stock is Rs. 160/- per month and there is a clause in the contract penalising him Rs. 100/- per engine per day late for missing the scheduled delivery date. Production of engines is in batches and each time a new batch is started, there is a setup cost of Rs. 10,000/- and lead time is 40 days. How frequently should batches be started and what should be the initial inventory level at the time each batch is started ? Also find the total minimum cost and reorder level. **[Delhi M.Sc. (OR.) 92]**

20.17. MODEL V : INVENTORY CONTROL SYSTEM

20.17–1. Fixed Order Quantity System

In this system of ordering a fixed quantity which is equal to the *Economic Order Quantity* (E.O.Q.) is ordered. The procurement and consumption cycle is graphically shown in *Fig.* 20·12.

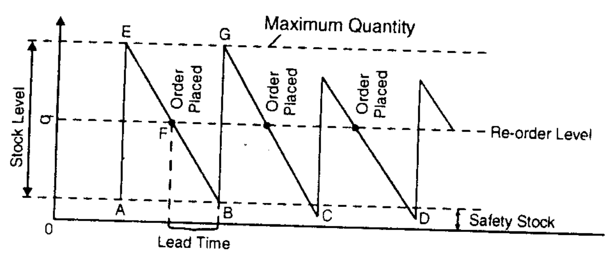


Fig. 20.12

From *Fig.* 20.12, it is clear that a supply equal to EOQ is received at the point A and the quantity in stock reaches a point E. The items are then issued and at time F the stock reaches the *re-order level* (R.O.L.), then the new order is placed for quantity q = E.O.Q, and the issues continued. At point B, the supplies of order placed at F are received when the inventory (stock) reaches G. At point G, there is a delay in receiving the supplies due to lead time and so we issue from the buffer stock. Thus for this system of ordering, we fix up the size of the order, *i.e.* every time the same quantity 'q' is ordered, but the time of placing an order is allowed to vary depending upon the actual demand (or usage). This is named as '*fixed order quantity*' or '*ordering level*' or '*two-bin system*'.

This is called ***'two-bin system'*** because the system is separated in *two* bins. One bin contains sufficient stock to meet the demand of time between the arrival of an order and placing the next. While the second bin contains sufficient stock to meet any probable demand during the period of replenishment. An order is placed as soon as the first bin is exhausted. So, in this system of ordering, we have to determine *three* parameteres for an item before placing an order. These are :

(i) E.O.Q. $(q^*) = \sqrt{2RC_3/C_1}$, *(ii)* Optimum Buffer Stock, and *(iii)* Re-Order Level (R.O.L.)

Useful formulae to remember :

(i) E.O.Q. $(q^*) = \sqrt{\dfrac{2RC_3}{C_1}}$,

(ii) Optimum Buffer (or Safety) Stock : B = (max. lead time – min. lead times) × R, where R is the consumption rate during lead time.

(iii) Re-Order Level (R.O.L.) = Buffer stock + Normal lead time consumption

(iv) Maximum Inventory = $B + q^*$, *(v)* Average Inventory = $B + \frac{1}{2} q^*$

(vi) Minimum Inventory = $B + 0 = B$.

Illustrative Examples

Example 29. *Consider the inventory system with the following data in usual notations :*
$R = 1000$ *units/year*, $I = 0.30$, P = *Rs.* 0.50 *per unit*, C_3 = *Rs.* 10.00, $L = 2$ *years (lead time).*
Determine : (i) *optimal order quantity,* (ii) *reorder point,* (iii) *minimum average cost.* **[JNTU 2002]**

Solution. Since we have $C_1 = IP = (0.30) \times (0.50)$ = Re. 0.15 per annum,

C_3 = Rs. 10.00, R = 1000 units/year.

(i) Optimal order quantity is given by the formula above, as

$$q^* = \sqrt{\left(\frac{2 \times 10 \times 1000}{0.15}\right)} = 365 \text{ units, and } t^* = \frac{q^*}{R} = \frac{365}{1000} = \frac{73}{200} \text{ or } 0.36 \text{ years.}$$ **Ans.**

(ii) Since the lead time is 2 years and optimum time is 0.36 years, reordering occurs when the level of inventory is sufficient to satisfy the demand for (2 – 0.36) years, *i.e.* for 1.64 years. Thus the optimum quantity $q^* = 365$ units is ordered when the order of inventory reaches to 1.64 × 1000 units. Therefore, *reorder point* = 1640 units.

(iii) Minimum average cost is given by

$$C_{\min} = \sqrt{2C_1C_3R} = (2 \times 0.15 \times 10 \times 1000) = 100\sqrt{0.30} = \text{Rs. } 54.8$$ **Ans.**

Example 30. *In a central grain store, it takes about 15 days to get the stock after placing an order, and daily 500 tons are despatched to neighbouring markets. On an ad hoc basis safety stock is assumed to be 10 day's stock. Calculate the reorder point p.* **[Rewa (M.P.) 93]**

Solution. We have B = 5,000 tons, L = 15 days, so that from the equation (20.86),

$$P = 5000 + (15 \times 500) = 12{,}500 \text{ tons.}$$ **Ans.**

Example 31. *A company uses annually 24000 units of a raw material which costs Rs. 1.25 per unit. Placing each order costs Rs. 22.5 and the carrying cost is 5.4 per cent per year of the average inventory. Find the economic order quantity, and the total inventory cost (including the cost of material).*

Should the company accept the offer made by the supplier of a discount of 5% on the cost price on a single order of 24,000 units ?

Suppose the company works for 300 days a year. If the procurement time is 12 days and safety stock is 400 units, find the re-order point, the minimum, maximum and average inventory.

Solution. Here we are given that

$$R = 24000 \text{ units/year}, C_3 = \text{Rs. } 22.5 \text{ per order}, C_1 = \text{Re. } 1.25 \times 5.4/100 = \text{Re. } 0.0675.$$

$$\therefore \quad q^* = \sqrt{\left(\frac{2C_3R}{C_1}\right)} = \sqrt{\left(\frac{2 \times 22.5 \times 24{,}000}{0.0675}\right)} = 4000 \text{ units.} \qquad \textbf{Ans.}$$

$t^* = q^*/R = 4000/24{,}000 = 1/6$ year $= 2$ months.

Therefore, the optimum lot size is 4000 units, after every 2 months. Total annual inventory cost

$= \sqrt{2C_1C_3R}$ + purchasing cost per year

$= \sqrt{2 \times 0.0675 \times 22.5 \times 24{,}000} + 1.25 \times 24{,}000 = 270 + 30{,}000 =$ Rs. 30,270. **Ans.**

Now in order to use the discount offer, if the company uses the size of 24,000 units, then each item costs Rs. $(0.95) \times 1.25$ and therefore, the fixed annual cost will be Rs. $(0.95) \times 1.25 \times 24{,}000 =$ Rs. 28,500.

Since he can order only once in a year, the annual ordering cost will be Rs. 22.50 and the average inventory is ½ (24,000).

So the annual carrying cost $= 1.25 \times 0.054 \times 0.95 \times 12{,}000 =$ Rs. 769.50.

$\therefore$ Associated annual cost = Rs. $(28{,}500 + 22.50 + 769.50) =$ Rs. 29,292.

Here we observe that the total cost is less than the total annual cost calculated above at the time of adopting the economic lot size policy, the discount offer is certainly profitable and so he will accept the offer. **Ans.**

Since the company works for 300 days a year, the demand for one day will be 24,000/300 units, *i.e.* 80 units.

So, in this case, $t^* = q^*/R = 4000/80 = 50$ days.

Since t^* is greater than the lead time and the safety stock is 400 units, the *re-order level* will be

= Safety stock + Normal lead time consumption $= 400 + 12 \times 80 = 1360$ units. **Ans.**

Average Inventory $= B + ½\, q^* = 400 + ½\,(4{,}000) = 2{,}400$ units. **Ans.**

Maximum Inventory $= B + q^* = 400 + 4000 = 4400$ units. **Ans.**

Minimum Inventory $= B + 0 = 400$ units. **Ans.**

Example 32. *For a fixed order quantity system, find out (i) Economic order quantity (ii) Optimum buffer stock (iii) Reorder level, for an item with the following data :*

Annual consumption $D = 10{,}000$ *units, Cost of one unit = Re.* 1.00, $C_3 =$ *Rs.* 12.00 *per production run* $C_1 =$ *Re.* 0.24 *per unit.*

Past lead times : 15 days, 25 days, 13 days, 14 days, 30 days, 17 days.

Solution.

(*i*) Economic order quantity $= \sqrt{\left(\frac{2C_3D}{C_1}\right)} = \sqrt{\left(\frac{2 \times 12 \times 10{,}000}{0.24}\right)} = 1{,}000$ units. **Ans.**

(*ii*) Optimum buffer stock = (Max. lead time – Normal lead time)× monthly consumption.

$$= \left(\frac{30 - 15}{30}\right) \times \frac{10{,}000}{12} = 416.66 \text{ units.} \qquad (\because \text{ normal lead time is 15 days})$$

For more safety, the optimum buffer stock may be rounded off to 420 units.

(*iii*) Normal lead time consumption = Normal lead time × monthly consumption

$= (15/30) \times (10{,}000/12) = 417$ units (approx.)

Therefore, the reorder level = Safety stock + Normal lead time consumption.

$= 420 + 417 = 837$ units or 840 units (approx.)

(*iv*) Maximum inventory level = Safety stock + Economic order quantity $= 420 + 1000 = 1420$ units.

Since inventory would fluctuate from a maximum of 1420 units to minimum of 420 units, the average inventory

= ½ [Safety stock + Max. Inventory] = ½ [420 + 1420] = 920 units.

Example 33. *A company uses annually 50,000 units of an item each costing Rs. 1.20. Each order costs Rs. 45 and inventory carrying costs 15% of the annual average inventory value.*

(i) *Find EOQ.* (ii) *If the company operates 250 days a year, the procurement time is 10 days and safety stock is 500 units, find the reorder level, maximum, minimum and average inventory.*

Solution. We are given the basic data :

$R = 50{,}000$ units, $P =$ Rs. 1.20, $C_3 =$ Rs. 45 per order, $C_1 = PI = 15\% \times$ Rs. 1.20.

(i) EOQ $= \sqrt{\dfrac{2C_3R}{C_1}} = \sqrt{\dfrac{2 \times 50{,}000 \times 45}{1.20 \times 0.15}} = 5{,}000$ units. **Ans.**

(ii) No. of days the company operates in a year = 250 days,
Lead time = 10 days, Safety Stock = 500 units,
Reorder level = Lead time demand + Safety stock

$= 10 \text{ days} \times \dfrac{50{,}000}{250 \text{ days}}$ units + 500 units = 10 days × 200 units + 500 units = 2,500 units.

Maximum Inventory = EOQ + Safety Stock = 5,000 units + 500 units = 5,500 units

Or

Maximum inventory = ROL + EOQ − Consumption during lead time
= 2,500 units + 5,000 units − 2,000 units 5,500 units.

Minimum inventory = Safety stock = 500 units,

Or

Minimum inventory = ROL − Consumption during lead time = 2,500 units − 2,000 units = 500 units.
Average inventory = (EOQ/2) + Safety Stock = (5,000/2) units + 500 units = 3,000 units.

Advantages and Disadvantages of System

Advantages : The advantages of this system are :

1. *It is simple, cheap to operate, and reliable.*
2. *It lends itself to operate and reliable.*
3. *It is preferable for one consumption value items.*
4. *It is appropriate for widely different types of inventory within the same firm.*
5. *It is most suitable when inventory carrying cost is measurable and significant.*
6. *There is automatic generation of replenishment order at appropriate time by comparison of stock level against re-order level.*
7. *It is somewhat more responsive to fluctuations in demand.*

Disadvantages : The main disadvantages of this system are :

1. *Many items may reach re-order level at the same time, thus overloading the re-ordering system.*
2. *There are no records of stock level and usage rates data.*
3. *It does not lend itself to ordering more items simultaneously from the same source.*

Remarks :

1. If lead time is not equal to zero and is known exactly, all that one needs to do is to order in advance by an amount of time equal to the lead time. When $q \le$ R.O.L., we purchase
$$(q^*) = \sqrt{2RC_3/C_1}.$$
2. It will be realised that some assumptions made here are not likely to be true in practice. For example, it seldom happens that the customer's demand is known exactly, or that the production time is negligible. But, the conceptual framework used in this model is sound.
3. The lead time is assumed to be sufficiently short so that not more than one order will remain in advance at any time.

Q. 1. Define the terms 'safety stock' and 'E.O.Q.' with the help of ideal inventory model.
2. Explain the problem of inventory control with deterministic demand.
3. Write on uses and abuses of maintaining inventories. **[Delhi M.Sc. (OR) 90]**

20.17–2. Periodic Review Inventory System

This system is also known as *(Fixed Interval System)* or *(Replenishment Inventory System)* or *(p-System)* or the (Order Cycle System). In this system, the size of order quantity may vary with the fluctuations in demand, but the ordering interval is fixed. This system is specified for any item by :

(i) Review period (t), and (ii) Requisitioning objective or replenishment level (r).

In this system, the inventory position is periodically reviewed regularly once *weekly/monthly/quarterly*/or *yearly*. If at any particular review the inventory position is say z, then order for $(r - z)$ units. Some safety stock would also be required to take care of any increased consumption or increase in lead time. The calculation of r is based on the formulae :

Replenishment level = Average consumption during a review period = Lead time + Safety stock
Order quantity = Replenishment level – Stock available.
'How the replenishment level obtained' can be explained from the cycle given below :

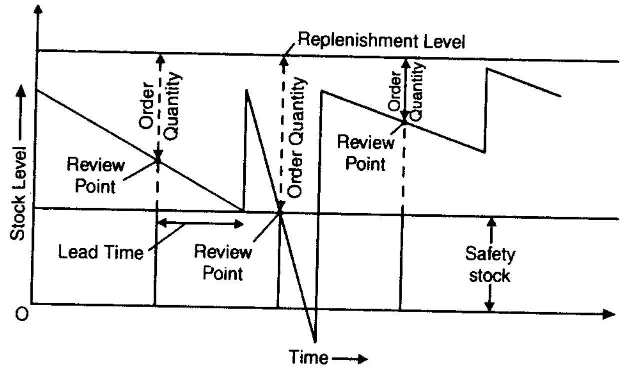


Fig. 20.13 Periodic Review System

In order to determine the review period for an item, the main considerations are :
(i) The minimum quantity on which a supplier should give good service (price, delivery, etc.)
(ii) The average consumption rate.

Advantages and Disadvantages of Periodic Review System

Advantages :

1. Since the inventory is reviewed periodically, there is no flexibility in the order period. Thus the fluctuation in demand is balanced by the safety stock.
2. Since all stock items are reviewed periodically, there is more chance of obsolete (old fashion) items being eliminated.
3. This system is preferable whenever inventory carrying cost is meaningless.
4. The larger quantity discounts may be obtained when a range of stock items are ordered at the same time from a supplier.
5. The ordering inventory costs are low. The suppliers give better price discounts since the sale is guaranteed.
6. Since the suppliers know your requirements in advance, the service is more efficient.
7. Since the orders will always be in the same sequence, there may be production economies due to more efficient production, planning and lower set up costs.

Disadvantages :

1. Since the system needs perpetual auditing, the cost of operating the system is higher.
2. In comparison to the fixed order system, this system requires more inventory in hand, for a given frequency of shortages.
3. It is less responsive to changes in consumption. If the rate of usage change shortly after a review, a stockout may certainly occur before the next review.
4. It is difficult to set appropriate periods for review, unless demands are reasonably consistent.
5. It needs additional labour to review items of stock at times other than when receipts and issues are being posted.

Illustrative Example

Example 34. *For a periodic review system, find out the various parameters for an item with the following data :*

Annual Consumption = 14,000 *units, Cost of One Unit* = *Rs.* 10, *Suppliers Min. Quantity* = 1,000 *units.*
Normal lead Time = 10 *days, Max. Lead Time* = 15 *days, Max. Consumption* = 1.20 (*average consumption*).

Solution. The maximum number of orders to cover the annual requirement would be 14,000/1,000 = 14 orders. Therefore, the review period should be 1/14th of an year or 26 days. But, for convenience, review period may be taken as one month instead of 26 days. Hence, *review period* = 1 month.

Safety stock = Max. Consumption rate (review period + max. lead time) – Normal Consumption rate (review period + normal lead time)

$$= 1.20 \times \frac{14000}{12}\left(1 + \frac{15}{30}\right) - \frac{14000}{12}\left(1 + \frac{10}{30}\right) = 2100 - 1555 = 545 \text{ units} \approx 550 \text{ units}$$

Replenishment level = Average rate of consumption × (review period + normal lead time) + safety stock

$$= \frac{14000}{12}\left(1 + \frac{10}{30}\right) + 550 \approx 2105 \text{ units}$$

Maximum inventory when the supplies are received

$$= 440 + \text{order quantity.} = 550 + 14000/12 \approx 1710 \text{ units.}$$

Minimum inventory would be = 550 units. Therefore, average inventory = ½ (1710 + 550) = 1130 units.

20.17–3. Hybrid System

In this system, the re-ordering point of *fixed order quantity system* and *variable order quantity of replenishment system* have been combined.

The Base-Stock Method or 3-Bin System is the most popular system. The items where *first bin* still contains some stock is not re-ordered although it belongs to a group with a common re-order period. If stock only remains in the *second bin*, however, the items are reordered, and when the stock remains only in *third bin*, a general review is adopted for the whole group of items.

EXAMINATION PROBLEMS

1. A particular item has an annual demand of 900 units. The carrying cost Rs. 2 per unit per year. The ordering cost per order is Rs. 90.

   (i) Find the Economic Order Quantity. *(ii)* Determine the number of orders to be placed per annum.

   (iii) If the purchase price per unit is Re. 0.50, what is the total cost per year ?

   (iv) If the lead time is two months and safety stock is 1000 units, what is the re-order point ?
   [Ans. (i) 900 units, (ii) 10, (iii) Rs. 6300, (iv) 2500 units.]

2. A company uses 10,000 units per year of an item. The purchase price is one rupee per item. Ordering cost = Rs. 25 per order. Carrying cost per year is 12% of the inventory value.

   (i) Find the EOQ, (ii) Find the number of orders/year, (iii) If the lead time is 4 weeks and assuming 50 working weeks per year, find the re-order point.
   [Ans. (i) 2083 units, (ii) 5 orders, (iii) 800 units.]

3. A factory uses Rs. 32,000 worth of a raw material per year. The ordering cost per order is Rs. 50 and the carrying cost is 20% per year of the average inventory.

   If the company follows the E.O.Q. purchasing policy, calculate the re-order point, maximum inventory, the minimum inventory, and the average inventory, it being given that the factory works for 360 days a year, the replenishment time is 9 days and the safety stock is worth Rs. 300.

   [Ans. R.O.L. = units worth Rs. 1100, maximum inventory = units worth Rs. 4300; min. inventory = units worth Rs. 300, average inventory = units worth Rs. 2300.]

4. In an inventory model, suppose that the shortages are not allowed and the production rate is infinite, and R = 600 units per year, I = 0.20, C_3 = Rs. 80.00, C = Rs. 3.00 and lead time is 1 year.

   (i) Find the optimal order quantity (ii) Re-order point (iii) Minimum average yearly cost.

   [Ans. (i) q^* = 400 units, (ii) R.O.L. = 200 units, (iii) C_{min} = Rs. 240.]

5. Calculate the various parameters for putting an item with following data on E.O.Q. system :

   Annual consumption is 12,000 units at the cost of Rs. 7.50 per unit. Set-up cost is Rs. 6 and the average inventory holding cost is Rs. 0.12 per unit. Normal lead time is 15 days and maximum lead time is 20 days.
   [Ans. R.O.L. = 367 units, Buffer stock = 200 units.]

6. Following information in an inventory problem is available : Annual Demand = 2400 units, Unit Price = Rs. 2.40, Ordering Cost = Rs. 4.00, Storage Cost = Rs. 2% per year, Interest Rate = 10% per annum, Lead Time = 15 days.

   Calculate : EOQ, reorder level, and total annual cost. How much does the total annual cost vary if the unit price is changed to Rs. 5 ?
   [Hint. Here R = 2400 units/year, C_3 = Rs. 4·00, C = Rs. 2.40, I = 2% + 10% = 12%, C_1 = CI = 0.2880]

   [Ans. (i) q^* = 258 units (ii) Re-order level = $\frac{1}{2} \times \frac{2400}{12}$ = 100 units,

(iii) Total inventory cost = min. variable cost + cost of 2400 units
= 74.36 + 5760 = Rs. 5834.36.

(iv) When C becomes Rs. 5, total inventory cost = Rs. 12,107.33.
$\therefore$ Increase in cost = Rs. (12,107.33 – 5834.36) = Rs. 6273.04.]

7. A company uses annually 48,000 units of raw material costing Rs. 1.20 per unit. Placing each order costs Rs. 45 and inventory carrying costs are 15% per year of the average inventory values.
   (i) Find EOQ, (ii) Suppose that the company follows the EOQ policy and it operates for 300 days a year, that the procurement time is 12 days and the safety stock is 500 units. Find the re-ordering level, the maximum, the minimum and average inventory.
   [**Hint.** Here R = 48,000 units, C = Rs. 1.20, I = 0.15, $C_1 = CI = 0.18$, C_3 = Rs. 45.]
   [**Ans.** (i) q^* = 4899 units (ii) Average daily demand = 48000/300 = 160 units/day
   Lead time damand = 12 × 160 = 1920 units, Safety Stock = 500 units.
   Re-order level = 1920 + 500 = 2420 units. Max. Inventory = 4899 + 500 = 5399 units.
   Min Inventory = Safety Stock = 500 units.
   Average Inventory = 1/2 q^* + Safety Stock = 2949 units]
8. The annual demand for a component is 7200 units. The carrying cost is Rs. 500/unit/year. The ordering cost is Rs. 1500/- per order and the shortage cost is Rs. 2000/unit/year. Find the optimal values of Economic Order Quantity, maximum inventory, maximum shortage capacity, and cycle time. **[JNTU (B. Tech.) 2003]**

III–Deterministic Models with Price Breaks

20.18. QUANTITY DISCOUNTS (PRICE-BREAKS) [Meerut 99]

So far we have described '*Elementary Deterministic Models*'. For these models, it was assumed that the unit production cost or purchase cost was constant over the range of possible order sizes. So there was no need to consider this cost directly. But, in this section, we shall consider a class of inventory problems in which such cost is a variable factor. However, to obtain a higher sales volume, many enterprises offer reduced purchasing costs for large purchases. Such discounts are typically referred to as *quantity discounts* or *price-breaks*. Given the chance to purchase large quantities on reduced price, the organization must decide between the *economic order quantity* ($q_E^* = \sqrt{2RC_3/C_1}$) and the quantity discount (q_D^*). The over all approach is to decide which option (q_E^* or q_D^*) minimizes the total costs, which must now include the purchasing cost in the cost equation (20.4). Thus, the objective here is : To minimize

total cost = Ordering cost + Holding cost + Purchasing cost, *i.e.*

$$C(q) = (R/q)\,C_3 + (q/2)\,PI + RP,$$

where the purchasing cost is usually obtained as annual demand R multiplied by the unit price (P), *i.e.* RP. We can decide whether or not to prefer the quantity discount by performing the following four steps :

Step. **1.** First compute q_E^*.

Step **2.** Then compute total cost using (q_E^*), *i.e.* $C(q_E^*)$.

Step **3.** Next compute total cost using (q_D^*), *i.e.* $C(q_D^*)$.

Step **4.** The optimal order quantity will be the quantity with the minimum cost.

For example, the schedule of *price breaks* may be given as follows :

Rs. 10.00 per item for any amount ordered upto a dozen (= 12 items)

Rs. 8.00 per additional item above a dozen and upto a gross (= 144 items)

Rs. 6.50 per additional item beyond a gross.

The mathematical expression for purchase cost corresponding to this schedule is given by

$$C(q) = \begin{cases} 10q & \text{for } 0 \le q \le 12 \\ 10 \times 12 + 8\,(q-12) & \text{for } 13 \le q \le 144 \\ 10 \times 12 + 8\,(144-12) + 6.50 \times (q-144) & \text{for } q \ge 145. \end{cases}$$

This situation is quite typical for purchased parts which are subject to quantity discounts. Though it is possible to develop this generalization of the inventory problem for each of the models discussed earlier, but here we shall discuss this generalization with respect to ***Model I*** only.

Since the production or purchase cost is variable, it must be considered directly. Hence, there will be a need for introducing some new symbols at this stage.

20.19. NOTATIONS USED

The variable costs associated with the production or purchasing process can be designated by—

P = cost per item of producing or purchasing

I = the cost of carrying one rupee in inventory value for one year expressed as a dimensionless fraction of one rupee

C_3 = set-up cost per production run or, when for purchased parts, the set-up cost associated with the attainment of the purchased items.

Further, as before, we let

R = number of items required per year (demand rate)

t = interval between placing orders

t^* = optimum interval between placing orders

q = input, or quantity ordered

q^* = optimum order quantity (*i.e.* Economic Lot Size or Economic Purchase Quantity).

Now we shall proceed to discuss *'purchase inventory model'* in general.

20.20. PURCHASE INVENTORY MODEL

Implied in this model are the following assumptions :

(i) demand rate R is constant

(ii) demand is both fixed and known

(iii) no shortages are to be permitted, consequently the cost of a shortage is infinite (i.e. using the notation of $C_2 \to \infty$)

(iv) the variable costs associated with the purchasing process are as mentioned above.

Determine : (a) How often should parts be purchased (t^*) ? (b) How many units should be purchased at any time (q^*) ?

Solution. First, we proceed to obtain basic cost equations as follows. Since, we have

$$q = Rt, \qquad \text{...(20.88)}$$

therefore, for each attainment, the number of part-month inventories will be given by

$$\tfrac{1}{2}\, qt = \tfrac{1}{2}\, q\,(q/R) = \tfrac{1}{2}\,(q^2/R)\,, \qquad \text{...(20.89)}$$

while the number of lot-month inventories will be given by

$$\tfrac{1}{2}\, qt/q = \tfrac{1}{2}\,(q^2/R)/q = \tfrac{1}{2}\,(q/R). \qquad \text{...(20.90)}$$

The component costs for each run will then be given by

C_3 = the attainment set-up cost,

qP = the purchasing cost of q items, where the unit purchase cost is given by P,

$C_3 \times \left(\frac{1}{2}\frac{q}{R}\right) \times I$ = cost (associated with the set-up) of inventory for period t,

$qP \times \left(\frac{1}{2}\frac{q}{R}\right) \times I$ = cost (associated with the purchase) of inventory for period t.

Therefore, the total cost for period t is given by

$$C_3 + qP + C_3 \,.\, \frac{1}{2}\frac{q}{R} \,.\, I + qP \,.\, \frac{1}{2}\frac{q}{R} \,.\, I.$$

Hence average cost per unit time is given by

$$C(q) = \frac{1}{t}\left(C_3 + qP + C_3 \,.\, \frac{1}{2}\frac{q}{R} \,.\, I + qP\,\frac{1}{2}\frac{q}{R}\, I \right)$$

or

$$C\,(q) = \frac{C_3 R}{q} + PR + \frac{C_3 I}{2} + \frac{qPI}{2} \quad (\text{since } t = q/R)$$

But, the term ½ C_3I being constant throughout the model may be neglected for the purpose of minimization or comparison of total costs.

$$\therefore \quad C(q) = C_3R/q + \tfrac{1}{2}\, qPI + PR. \qquad \text{...(20.91)}$$

The minimum of $C(q)$ can be obtained by taking the first derivative of $C(q)$ with respect to the variable q and setting the resulting expression equal to zero, *i.e.*

$$\frac{dC(q)}{dq} = -\frac{C_3R}{q^2} + \frac{1}{2}PI.$$

Therefore, setting $dC(q)/dq = 0$, we get

$$q^* = \sqrt{\left(\frac{2C_3R}{IP}\right)}. \qquad \text{...(20.92)}$$

Substituting the value of q^* in equation (2.91), the optimum total cost $C(q^*)$, associated with a unit purchase cost P is obtained as

$$C(q^*) = \frac{C_3R}{\sqrt{2C_3R/PI}} + \frac{PI}{2}\sqrt{\left(\frac{2C_3R}{PI}\right)} + P.R$$

$$= \sqrt{\left(\frac{C_3RPI}{2}\right)} + \sqrt{\left(\frac{C_3PRI}{2}\right)} + P.R$$

i.e.
$$C(q^*) = \sqrt{2C_3RPI} + PR. \qquad \text{...(20.93)}$$

Q. 1. Formulate and solve a mathematical model for all units discounts when shortages are not allowed to obtain the optimal value of the order quantity. **[JNTU (MCA III) 2004; Raj. Univ. (M. Phil) 93]**

2. Distinguish between 'All Units Discounts' and 'Incremental Discounts'. Formulate a mathematical model for all units discounts when shortages are allowed and backlogged fully. Obtain the optimum value of order quantity. **[JNTU (MCA II) 2004; Delhi M. Sc. (OR) 92]**

3. Explain the relevant costs for inventory decisions. How are these costs sought to be controlled with the O.R. techniques ?

4. With the help of quantity-cost curve, explain the significance of economic order quantity. What are the limitations in using economic order quantity formula ? **[JNTU (Mech. & Prod.) May 2004]**

Illustrative Example

Example 35. *A Purchase Manager has decided to place order for a minimum quantity of 500 units of a particular item in order to get a discount of 10%. From the records, it was found that in the last year 8 orders each of size 200 units have been placed. It is given that ordering cost = Rs. 500 per order, inventory carrying cost = 40% of the inventory value, and the cost per unit = Rs. 400. Is the Purchase Manager justified in his decision ? What is the effect of his decision to the company ?* **[JNTU (MCA III) 2004]**

Solution. In this problem, we are given that—

$R = 8 \times 200$ units per year, C_3 = Rs. 500 per order, P = Rs. 400 per unit, I = 40% of inventory value.

(i) If q^* is the optimum order quantity, then

$$q^* = \sqrt{\frac{2C_3R}{PI}} = \sqrt{\frac{2 \times 500 \times 1600}{400 \times 0.4}} = 100 \text{ units.}$$

Total inventory cost as per EOQ approach : $C(q^*) = RP + \dfrac{RC_3}{q^*} + \dfrac{1}{2}q^*\,PI$

$$\therefore \quad C(100) = 1600 \times 400 + \frac{1600}{100} \times 500 + \tfrac{1}{2} \times 100 \times (400 \times 0.4) = \text{Rs. } 656{,}000. \qquad \text{...(1)}$$

(ii) Total cost as per present policy (i.e. for order size of 200 units) :

$$C(200) = 1600 \times 400 + \frac{1600}{200} \times 500 + \tfrac{1}{2} \times 200 \times (400 \times 0.4) = \text{Rs. } 6{,}60{,}000. \qquad \text{...(2)}$$

(iii) Proposed inventory costs : If the manager decides to place an order for a minimum quantity of 500 units to avail a discount of 10%, the cost per item becomes Rs. 400 × 0.9 = Rs. 360.

$$\therefore \quad C(500) = 1600 \times 360 + \frac{1600}{500} \times 500 + \frac{1}{2} \times 500 \times (360 \times 0.4) = \text{Rs. } 6{,}13{,}600. \qquad \text{...(3)}$$

From (1), (2) and (3) above, it is quite clear that if the manager decides to place an order for a minimum quantity of 500 items, he will save Rs. (6,60,000 – 6,13,600) = Rs. 46,400 over the present policy of placing an order of size 200 units. Also, he will save Rs. (6,56,00 – 6,13,600) = Rs. 42,400 over the policy of placing an order of economic size 100 units.

Hence, the purchase manager's decision is justified and Rs. 46,400 will be saved.

EXAMINATION PROBLEMS

1. A company buys 8,000 units of an item for its annual requirements. Each unit costs Rs. 10. The ordering cost for order is Rs. 30 and the carrying cost is 7.5% of the average inventory per year.
   (i) Determine the EOQ and the total inventory cost.
   (ii) Should the company accept an offer of 2% discount in price on 4 bigger orders of quarterly requirements of the material ?
   [**Ans.** $q^* = 800$ units, C_{min} = Rs. 80,600]
2. A publishing house purchases 2,000 units of a particulars item per year at unit cost of Rs. 20, the ordering cost per order is Rs. 50 and the inventory carrying cost is 25 per cent. Find the optimum order quantity and minimum total cost including purchase cost. If a 3% discount is offered by the supplier for the purchase in lots of 1,000 or more, should the publishing house accept the proposal.
   [**Ans.** No.]
3. To decide to buy an item the following data are given :
   Annual demand = 600 units, Ordering cost = Rs. 400, Holding cost = 40%, Cost per units = Rs. 15, Discount is 10% if the order quantity = 500.
   What should be the decision ? Justify your answer.
   [**Ans.** The decision should be to place an order of size 500 units and avail the 10% discount, because there will be a saving of Rs. (10,697 – 9,930) = Rs. 767.]
4. A company uses 8,000 units of a product as raw material, costing Rs. 10/– per unit. The administrative cost per purchase is Rs. 40/–. The holding costs are 28% of the average inventory. The company is following an optimum purchase policy and places orders according to the EOQ. If has been offered a quantity discount of one per cent if it purchases its total annual requirement only four times a year.
   Should the offer be accepted ? If not, what discount should the company demand ?
5. Annual demand for a particular item of inventory is 10,000 units. Inventory carrying cost per year is 20% and ordering cost is Rs. 40 per order. The price quoted by the supplier is Rs. 4 per unit. However, the supplier is willing to give a discount of 5% for orders of 1,500 or more. Is it worth while to avail of the discount offer.
   [**Ans.** Yes.]
6. A materials manager has the following data for procuring a particular item. Annual demand = 1000, Ordering cost = Rs. 800/–, Inventory carrying cost = 40%, Cost per item = Rs. 60. If the order quantity is more than or equal to 300, a discount of 10% is given. For how much should he place the order in order to minimize the total variable cost ?

20.21. PURCHASE INVENTORY MODEL WITH ONE PRICE BREAK

In this section, we shall consider such type of purchasing situation where only *one quantity discount* applies. Such type of situation may be represented as follows :

Purchase cost (P) per item	***Range of quantity***
P_1	$1 \le q_1 < b$
P_2	$q_2 \ge b$

where b is the quantity at and beyond which the quantity discount applies. Ovbiously, $P_2 < P_1$.

Thus, for any purchase quantity q_1, in the range $1 \le q_1 < b$, the total expected cost $C(q_1)$ will be given by

$$C(q_1) = \frac{C_3 R}{q_1} + P_1 R + P_1 \cdot \frac{q_1 I}{2} \quad \text{[using eqn.(20.91)]} \qquad ...(20.94)$$

Similarly, for any purchase quantity q_2, in the range $q_2 \ge b$, the total expected cost will be given by

$$C\,(q_2) = \frac{C_3 R}{q_2} + P_2 R + P_2 \cdot \frac{q_2 I}{2} \cdot \qquad ...(20.95)$$

The graphical representation of this situation will be as follows :

First, if we neglect, for the moment, the term $[P_1 R\,]$ in equation (20.94) and the term $[P_2 R]$ in equation (20.95), we shall obtain *Fig.* 20.14. The graphical representation for complete equations (20.94) and (20.95) can be seen in *Fig.* 20.16. Since $P_2 < P_1$, we have $P_2 R \; < \; P_1 R$.

Furthermore, from *Fig. 20.14* [and also from equations (20.94) and (20.95)], it is clear that the minimum cost for curve II (corresponding to P_2) is less than the minimum cost for curve I (corresponding to P_1), that is,

$$C(q_2^*) < C(q_1^*) \quad \text{[from (20.93)]}$$

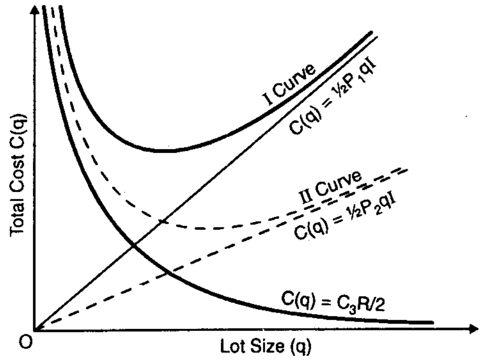


Fig. 20.14 Economic lot size curves : one price break.

In general, for n price breaks, the following inequalities hold : $q_1^* < q_2^* < q_3^* < \ldots < q_n^*$.

They follow immediately by considering equation (20.93) and $P_n < P_{n-1} < P_{n-2} < \ldots < P_1$.

We now derive the following decision rules.

Rule 1. Compute q_2^* by using formula (20.92). If $q_2^* \geq b$ as shown in the following figure, then the previous discussion applies and the optimum lot size will be q_2^*.

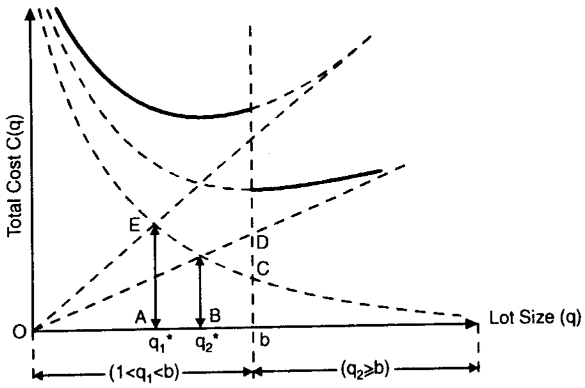


Fig. 2.15 Economic lot size, $q_2^* \geq b$

Rule 2. If $q_2^* < b$ as shown in the following figure, then the quantity discount no longer applies to the purchase quantity q_2^*.

Furthermore, the minimum cost occurs at a point for which the abscissa is *less* than b, that is, at $q_2^* < b$, so the total expected cost will be monotonic increasing over the entire price range ($q_2 \geq b$), and the minimum cost for range ($q_2 \geq b$) will occur at $q = b$. Hence to determine the optimum purchase quantity, we only need to compare the total expected cost for lot size $q = q_1^*$ with that for lot size $q = b$. These cost equations follow from equation (20.91) and are given by

$$C(q_1^*) = \frac{C_3 R}{q_1^*} + \frac{P_1 I q_1^*}{2} + P_1 R \quad \text{...(20.96a)}$$

$$C(b) = \frac{C_3 R}{b} + \frac{P_2 I b}{2} + P_2 R \quad \text{...(20.96b)}$$

Now comparing the sum of first and third terms of $C(b)$ and $C(q_1^*)$ it shows that

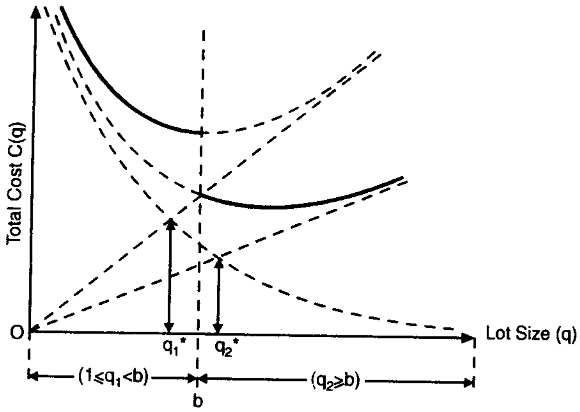


Fig. 20.16 Economic lot size curve, q_2* < b

$$\frac{C_3R}{b} + P_2R < \frac{C_3R}{q_1^*} + P_1R \quad \text{(because } q_1^* < b \text{ and } P_2 < P_1\text{)}$$

However, ½ P_2Ib may or may not be less than the corresponding term ½ $P_1Iq_1^*$. Hence, we must compare the total cost as indicated in the foregoing.

If $C(q_1^*) < C(b)$ then q_1^* is the optimum order quantity, otherwise b is the optimum order quantity.

From *Fig.* 20.16, we observe that we are adding (P_1R) to $\overline{2EA}$ and compare this value with $[P_2R + \overline{DB} + \overline{CB}]$. Now see *Fig.* 20.17.

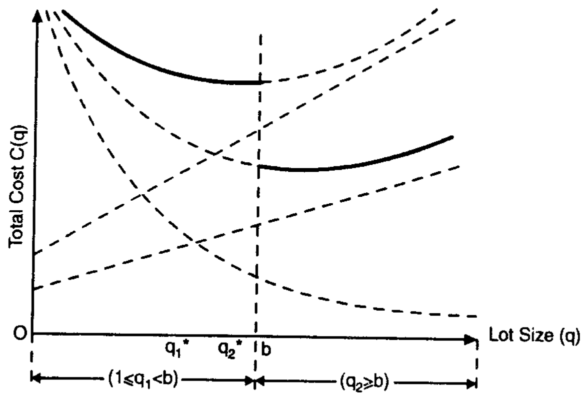


Fig. 20.17 Economic lot size curve, q^* = b

Q. 1. Formulate and solve the purchase inventory problem with one price break.

2. Formulate a mathematical model for incremental unit discount when shortages are not allowed, find also the optimal value of the order quantity. **[Delhi M.Sc. (OR.) 90]**

The use of above two decision rules can best be explained in the following numerical examples.

20.21-1. Illustrative Examples

Example 36. *Find the optimum order quantity for a product for which the price breaks are as follows :*

Quantity	*Unit Cost (Rs.)*
$0 \le q_1 < 500$	10.00
$500 \le q_2$	9.25

The monthly demand for a product is 200 units, the cost of storage is 2% of unit cost and the cost of ordering is Rs. 350.

Solution. It is given that

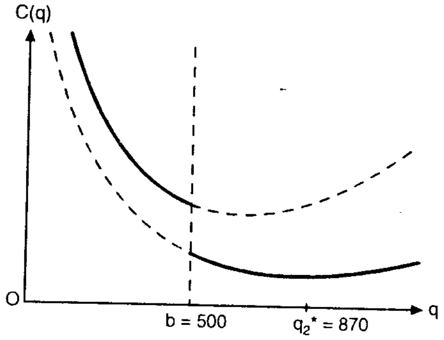


Fig. 20.18. $q^* = 870$

$R = 200$ units/month, $I =$ Re. 0.02, $C_3 =$ Rs. 350, $P_1 =$ Rs. 10.00, $P_2 =$ Rs. 9.25.

Using equation (20.92), compute q_2^* and get

$$q_2^* = \sqrt{\left(\frac{2C_3R}{P_2I}\right)} = \sqrt{\frac{2 \times 350 \times 200}{9.25 \times 0.02}} = 870 \text{ units (nearly)}$$

Here $q_2^* > b$, *i.e.* $870 > 500$. Since $q_2^* = 870$ lies within the range $q_2 \geq 500$, the optimum purchase quantity will be $q^* = 870$ *(by Rule 1)*. This situation can be represented graphically in *Fig. 20.18.*

Example 37. *Solve Example 36 when the procurement setup cost C_3 is only Rs. 100 (instead of Rs. 350).*

[JNTU (B. Tech.) 2002]

Solution. As in *Example* 36, compute

$$q_2^* = \sqrt{\frac{2C_3R}{P_2I}} = \sqrt{\frac{2 \times 100 \times 200}{9.25 \times 0.02}} = 465 \text{ units}$$

Since $q_2^* < b$ $(\because 465 < 500)$, apply *Rule 2* and compute

$$q_1^* = \sqrt{\left(\frac{2C_3R}{P_1I}\right)} = \sqrt{\frac{2 \times 100 \times 200}{10 \times 0.02}} = 447. \text{ units}$$

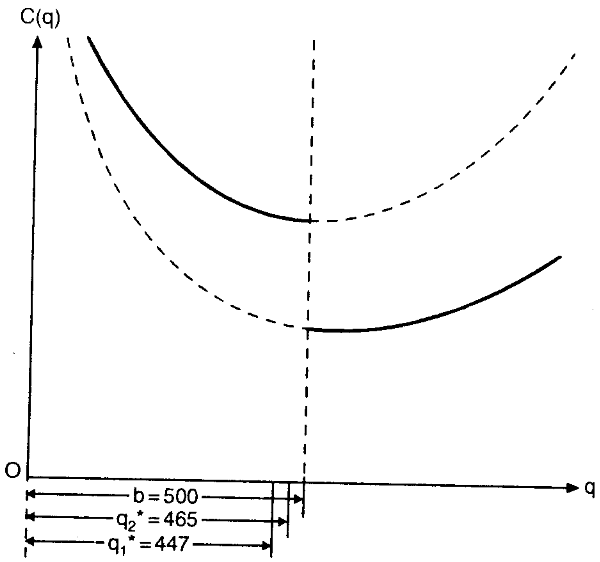


Fig. 20.19. $q^* = 500$

Then compare $C(447)$ with $C(500)$, *i.e.* the optimum cost of procuring the least quantity which will entitle for price break (in this case, $b = 500$).

Therefore, from equations (20.91) and (20.93),

$$C(q_1^*) = C(447) = \sqrt{(2 \times 10 \times 0.02 \times 100 \times 200)} + 10 \times 200 = \text{Rs. } 2090.42\,. \qquad \ldots(20.97)$$

$$C(b) = C(500) = \frac{100 \times 200}{500} + 9.25 \times 200 + \frac{1}{2} \times 9.25 \times 0.02 \times 500\; C(500) = \text{Rs. } 1937.25 \ldots(20.98)$$

Since $C(b) < C(q_1^*)$ or $C(500) < C(447)$, from (20.97) and (20.98), the optimum purchase quantity is determined 500 by comparing the price. Hence $q^* = b = 500$. **Ans.**

This situation is shown graphically in *Fig.* 20.19.

Example 38. *Solve Example 26 when C_3 = Rs.* 100 *(instead of Rs. 350), b =* 3000 *(instead of Rs. 500).*

Solution. Here, $q_2^* = 465 < 3000$. Therefore, compute q_1^* which is previously determined as equal to 447.

Also, from *Example 3* result (20.97)

$$C(q_1^*) = C(447) = \text{Rs. } 2090.42 \qquad \ldots(20.99)$$

To Compare $C(q_1^*)$ with $C(b)$, *i.e.* $C(447)$ with $C(3000)$, compute

$$C(3000) = \frac{100 \times 200}{3000} + 9.25 \times 200 + \frac{1}{2} \times 9.25 \times 0.02 \times 3000 = \text{Rs. } 2135.17. \qquad \ldots(20.100)$$

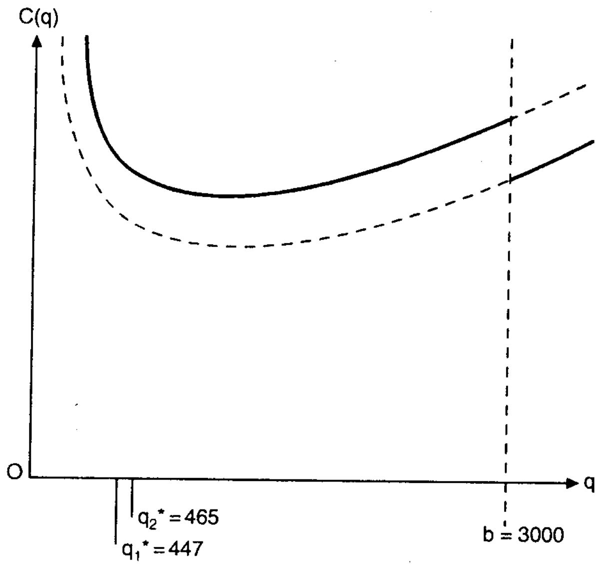


Fig. 20.20. $q^* = 447$

Here, from (20.99) and (20.100) we observe that $C(3000) > C(447)$

Therefore, the optimum purchase quantity is $q^* = q_1^* = 447$. units. **Ans.**

This situation is represented graphically in Fig. 20.20.

Example 39. *A company uses annually 24,000 units of raw material which costs Rs. 1.25 per unit. Placing each order costs Rs. 22.50 and the carrying cost is 5.4% per year of the average inventory. Find the economic lot size and the total inventory cost (including cost of material).*

Should the company accept the offer made by the supplier of a discount of 5% on the cost price on a single order of 24000 units.

Solution. Given that $R = 24000$ units, C_3 = Rs. 22.50 per order, P = Rs. 1.25 per unit, and

I = Inventory carrying cost = 5.4% per year of average inventory (= 0.054).

Optimum order quantity $q^* = \sqrt{\left(\frac{2C_3R}{PI}\right)} = \sqrt{\frac{2 \times 22.50 \times 24{,}000}{0.054 \times 1.25}} = 4000$ units.

The optimum number of orders in a year $= R/q = 24000/4000 = 6$ per year.

Total ordering cost = Rs. (6 × 22.50) = Rs. 135.00,

Annual inventory carrying cost = Rs. [½ (4000) (0.054) × 1.25] = Rs. 135.00.

Cost of procuring 24,000 units = Rs. 1.25 × 24000 = Rs. 30,000.

Thus the total annual minimum cost = Rs. [135 + 135 + 30,000] = Rs. 30,270.

If the company wishes to avail the discount offer, then the reduced price of each unit at the rate of 5% discount will be Rs. 0.95 × 1.25.

The cost of 24,000 units, therefore, will be Rs. 0.95 × 1.25 × 24,000 = Rs. 28,500.

Now when there is only one order of 24000 units, the storage cost is

Rs. 0.054 × 1.25 × 0.95 × 24,000 = Rs. 769.50

Therefore, the total annual cost when 24,000 units are purchased in a single order with 5% discount

= Rs. [28,500 + 22.50 + 769.50] = Rs. 29,292.

Since the total annual cost with discount of Rs. 29,292 is less than the total annual cost without discount of Rs. 30270, the company should accept the offer of 5% discount in place of using an EOQ policy of 4000 units per order.

Example 40. (*a*) *The soft goods department of a large departmental store sells 500 units per month of a certain large bath towel. The unit cost of a towel to the store is Rs. 10 and the cost of placing an order has been estimated to be Rs. 50. The store uses an inventory carrying charge of 20% of average inventory valuation per month. Assuming that the demand is deterministic and continuous, and that no stockouts are allowed; determine the optimal order quantity. What is the time between placing of orders ? The procurement lead time for the towels is one month. What is the recorder point based on the on-hand inventory level ?*

(*b*) *Consider an item on which incremental quantity discounts are available. The first hundred units cost Rs. 100 each and additional units cost Rs. 95 each. For this item, demand* = 500 *units per year, inventory carrying charge* = 20% *of average inventory valuation per annum, and procurement cost* = *Rs.* 50. *Determine, the EOQ.*

Solution. (a) Here $R = 500$ units per month, $C_3 = Rs.\ 50.00$, $C_1 =$ Rs. (10×0.20) per month = Rs. 2.00 per month.

$$\therefore \quad q^* = \sqrt{\left(\frac{2C_3R}{C_1}\right)} = \sqrt{\left(\frac{2 \times 50 \times 500}{2}\right)} = 158 \text{ units of towels}$$

and $$t^* = q^*/R = 158/500 = 0.316 \text{ month.}$$

Since the lead-time is given one month, reordering occurs when the level of inventory is sufficient to satisfy the demand for (1 – 0.316) month. Therefore, reorder point = (1 – 0.316) × 500 = 342 units.

(**b**) Given that $R = 500$ units per year; $C_3 =$ Rs. 50.00 and $I =$ Rs. 0.20 per year.

Quantity discounts are given below ;

Unit cost	*Quantity*
Rs. 100.00	$0 \le q_1 \le 100$
Rs. 95.00	$100 \le q_2$

Now $$q_2^* = \sqrt{\left(\frac{2C_3R}{P_2I}\right)} = \sqrt{\left(\frac{2 \times 50 \times 500}{95 \times 0.20}\right)} = 100\sqrt{\left(\frac{5}{19}\right)} = 51.3 \text{ units.}$$

Since $q_2^* < b$ (= 100), we next compute q_1^* to have

$$q_1^* = \sqrt{\left(\frac{2C_3R}{P_1I}\right)} = \sqrt{\left(\frac{2 \times 50 \times 500}{100 \times 0.20}\right)} = 50 \text{ units.}$$

We now compute the costs :

$$C\,(q_1^*) = 50 \times \frac{500}{50} + 500 \times 100 + 100 \times 0.20 \times \frac{50}{2} = \text{Rs. } 51{,}000$$

$$C(b) = 50 \times \frac{500}{50} + 500 \times 95 + 95 \times 0.20 \times \frac{100}{2} = \text{Rs. } 48{,}700$$

Since $C(b) < C(q^*)$, the optimum order quantity is $q^* = b = 100$ units. **Ans.**

EXAMINATION PROBLEMS

1. Find the optimum order quantity for a product for which the price breaks are given below :

Quantity	:	$0 \le q_1 \le 500$	$q_2 \ge 500$
Unit Cost (Rs.)	:	15.00	14.50

Monthly demand for the product is 250 units, cost of storage is 2% of the unit cost and cost of ordering is Rs. 300.
[**Ans.** $q^* = 707$ units.]

2. The requirement for a type of boiler bolt is 2000 per year. The purchase price is quoted at Rs. 10 per unit in quantities below 1000, and Rs. 9.50 per unit in quantities above 1000. Ordering costs are Rs. 1000 per order and inventory costs are 16 per cent per year per unit of average inventory value. Determine the optimum purchase schedule.
[**Ans.** $q^* = 500$ units]

3. The annual demand of a product is 10,000 units. Each unit costs Rs. 100 if orders placed in quantities below 200 units, but for orders of 200 or above the price is Rs. 95. The annual inventory holding cost is 10% of the value of the item and the ordering cost is Rs. 5.00 per order. Find the economic lot size.
[**Hint.** Here $R = 10{,}000$, $P_1 =$ Rs. 100, $P_2 =$ Rs. 95, $I =$ Rs. 0·10, $C_3 =$ Rs. 5.00, $q_2^* = 103$, $q_1^* = 100$, $C(100) < C(200)$]
[**Ans.** $q^* = 100$ units.]

4. A sport company buys 2000 bats annually. A fixed cost of Rs. 50 is incurred each time an order is placed. Inventory carrying charge is estimated at 20%. Supplier offers a 10% discount in price per bat of Rs. 100 if orders are placed for more than or equal to 150 bats at a time. In what order size should the company purchase.
[**Ans.** $q^* = 150$ bats.]

5. An item used in the manufacture of surgical equipment has a steady annual usage rate of 1200 dozen. Cost of the item is Rs. 3 per dozen. The manufacturer considers his cost of ordering these parts to be Rs. 5 per order and this annual cost of stored inventory to be 10%. Determine the optimal lot size. If the surgical supply manufacturer is offered 10% discount on his purchases in case he orders no more than twice per year, should he agree to this ?
[**Ans.** $q^* = 200$ units, Yes, he should agree.]

6. The estimated annual demand for a product is 5,000 units, the ordering cost is Rs. 49 per order and the inventory carrying cost (per year) is 20% of the value of inventory. The normal price charged by the supplier is Rs. 5 per unit. However, a discount @3% is allowed for an order for atleast 100 units. The discount is raised to 5% if the order is for 2500 units or more. What is the most economical order quantity ?

7. Find EOQ for the following :
Annual demand = 400 units, Ordering cost = Rs. 20, Inventory carrying charges = 20%,
Cost per unit is Rs. 50 if $q < 100$, and Rs. 49 if $q \ge 100$. (q = order quantity).

8. The annual demand for a product is 64,000 units (or 1280 units per week). The buying cost per order is Rs. 10 and the estimated cost of carrying one unit in stock for a year is 20%. The normal price of the product is Rs. 10 per unit. However, the supplier offers a quantity discount of 2% on an order of at least 1000 units at a time and a discount of 5% if the order is for at least 5,000 units.
Suggest the most economic purchase quantity per order.

9. Find the optimal economic order quantity for a product having the following characteristics :
Annual demand = 2,400 units, Ordering cost = Rs. 100, Cost of storage = 24% of unit cost

Price break :	*Quantity*	:	$0 \le q \le 500$	$500 \le q$
	Unit cost	:	10.00	9.00.

[**Ans.** 472 units.]

10. The annual demand for a product is 10,000 units. The order-processing cost is Rs. 60/– per order and the carrying costs are 2½% per month. The normal price of the product is Rs. 10/– per unit. However, the supplier offers a quantity discount of 5% on an order of 600 units and a discount of 10% on an order of 2000 units.
Suggest the most economic purchase quantity. **[Delhi (M.Com.) 90]**

11. Consider an item an which incremental quantity discounts are available. The first 10 units cost Rs 100/- each and additional units cost Rs. 95/- each.
Determine the optimal order quantity, when $\lambda = 500$ units per year, $I = 0{\cdot}20$, $A =$ Rs. 50/- per setup.
[Delhi M.Sc. (OR) 90]

20.22. PURCHASE-INVENTROY MODEL WITH TWO PRICE BREAKS

In this section, we generalize one step further by considering a purchasing situation when *two quantity discounts* apply. Such a situation may be represented as follows :

Purchase cost P per item	*Range of quantity*
P_1	$1 \le q_1 < b_1$
P_2	$b_1 \le q_2 < b_2$
P_3	$b_2 \le q_3$

where b_1 and b_2 are the quantities which determine the price breaks.

In this case also, the same general discussion holds as discussed for one price break only. Thus, to obtain the optimum purchase quantity following *Working Rule* is taken into practice.

Working Rule :

***Step* 1.** Compute q_3^* and compare with b_2.

(i) If $q_3^* \ge b_2$, then the optimum purchase quantity is q_3^*, *(ii)* If $q_3^* < b_2$, then go to *Step* 2.

***Step* 2.** Compute q_2^*. Since $q_3^* < b_2$ and q_2^* is also less than b_2 (because $q_1^* < q_2^* < q_3^* < \ldots < q_n^*$, in general). Thus, there are only two possibilities when $q_2^* < b_2$, *i.e.* either $q_2^* \ge b_1$ or $q_2^* < b_1$.

(i) If $q_2^* < b_2$ but $\ge b_1$, then proceed as in the case of one price break only. That is, compare the costs $C(q_2^*)$ and $C(b_2)$ to obtain the optimum purchase quantity. The quantity with lower cost will naturally be the optimum.

(ii) If $q_2^* <$ (b_2 and b_1 both), then go to *Step* 3.

***Step* 3.** If $q_2^* <$ (b_2 and b_1 both), then compute q_1^* which will satisfy the inequality $q_1^* < b_1$. In this case, compare the cost $C(q_1^*)$ with $C(b_1)$ and $C(b_2)$ both to determine the optimum purchase quantity.

Now the application of above decision rules are explained with the help of five numerical examples in different situations.

20.22–1. Illustrative Examples

Example 41. *Find the optimal order quantity for a product for which the price breaks are as follows :*

Quantity :	$0 \le q_1 < 500$	$500 \le q_2 \le 750$	$750 \le q_3$
Unit cost (Rs.) :	10.00	9.25	8.75

The monthly demand for a product is 200 units, the cost of storage is 2% of the unit cost and the cost of ordering is Rs. 350. **[Meerut 99; Agra 98]**

Solution. It is given that—

$R = 200$ units/month, $C_3 =$ Rs. 350, $I = 0.02$, $P_1 =$ Rs. 10.00, $P_2 =$ Rs. 9.25, $P_3 = 8.75$, $b_1 = 500$, $b_2 = 750$.

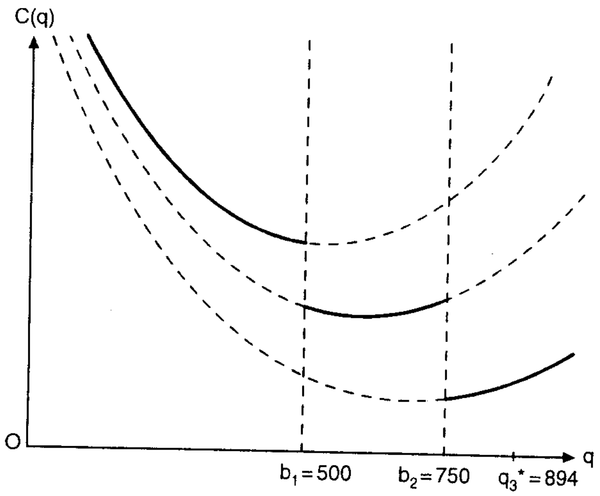


Fig. 20.21. $q^* = 894$.

Step 1. Compute q_3^* and obtain, $\quad q_3^* = \sqrt{\left(\dfrac{2C_3R}{P_3I}\right)} = \sqrt{\left(\dfrac{2 \times 350 \times 200}{8.75 \times 0.02}\right)} = 894.$

Since $q_3^* > b_2$, *i.e.* $894 > 750$, therefore the optimum purchase quantity will be $q^* = 894$. **Ans.**

This situation is explained graphically in *Fig.* 20.21.

Example 42. *Find q^*, where $R = 200$ items/month, $C_3 = Rs.$ 100, $I = 0.02$ (i.e. 2% of the unit cost) and for $P_1 = Rs.$ 10.00 for $0 \le q_1 < 500$, $P_2 = Rs.$ 9.25 for $500 \le q_2 < 750$, $P_3 = Rs.$ 8.75 for $q_3 \ge 750$.*

[Meerut 2005]

Solution. Compute q_3^* and obtain

$$q_3^* = \sqrt{\left(\frac{2 \times 100 \times 200}{8.75 \times 0.02}\right)} = 478. \text{ units}$$

Since $q_3^* < b_2$, *i.e.* $(478 < 750)$, so compute q_2^*. This has already been calculated in ***Example* 37**, *i.e.* $q_2^* = 465$ units.

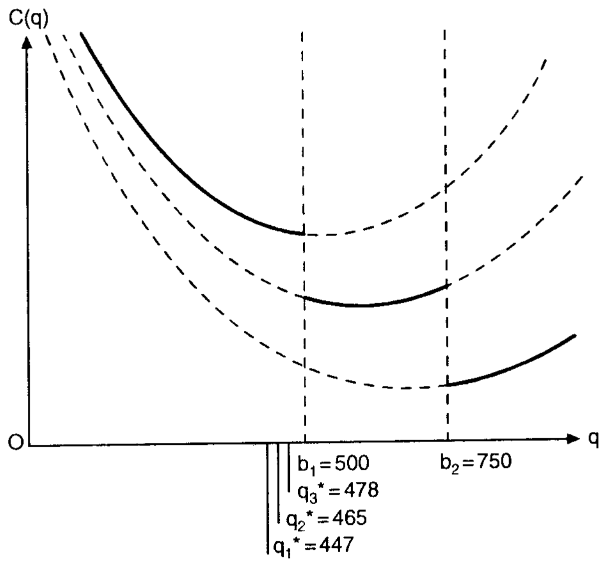


Fig. 20.22. $q^* = 750$

Since $q_2^* < b_1$, *i.e.* $(465 < 500)$, compute q_1^* which is given by $q_1^* = 447$ units ***(Example* 37).**

Now, we must compare C (447) with C (500) and C (750).

From ***Example* 37,** we have $C(q_1^*) = C(447) = \text{Rs. } 2090.42$, $C(b_1) = C(500) = \text{Rs. } 1937.25$.

Furthermore, from equation (20.91), $C(b_2) = C(750) = \text{Rs. } 1843.29$.

Therefore, $C(750) < C(500) < C(q_1^* = 447)$, the optimum purchase quantity is $q^* = 750$.

This situation is shown graphically in *Fig.* 20.22 above.

Example 43. *This example is same as **Example 37** except that $b_1 = 400$ (instead of 500) and $b_2 = 3000$ (instead of 750).* **[Meerut (M.Sc.) 96]**

Solution. As in ***Example* 37,** $q_3^* = 478$ and $q_2^* = 465$.

Since $q_2^* = 465$ falls within the range $(400 \le b_2 < 3000)$, there is no need to calculate q_1^*. Rather, compare only $C(q_2^*)$ with $C(3000)$.